EMSE 4765: DATA ANALYSIS
For Engineers and Scientists

Session 14: Two-Way Analysis of Variance (ANOVA),
2^K-Factorial Analysis of Variance (ANOVA)

Version: 4/20/2020

Lecture Notes by: J. René van Dorp[1]
www.seas.gwu.edu/~dorpjr

[1] Department of Engineering Management and Systems Egineering, School of Engineering and Applied Science, The George Washington University, 800 22nd Street, N.W., Suite 2800, Washington D.C. 20052. E-mail: dorpjr@gwu.edu.

Aircraft Primer Paint Example:

Air craft primer paints are applied to aluminum surfaces by two methods – dipping and spraying. The purpose of the primer is to improve paint adhesion; some parts can be primed using either application method. An engineer interested in whether three different primers differ in their adhesion properties performed an experiment to investigate the effect of paint primer type and application method on paint adhesion. Three specimens were painted with each primer using each application method, a finish paint was applied, and the adhesion force was measured. The

$$3 \cdot 2 \cdot 3 \text{ experiments} = 18 \text{ experiments}$$
$$(\#\text{Primer Types}) * (\#\text{Application Methods}) * (\#\text{Replications})$$

from this experiment were run in a random order. The resulting data is shown in the Table below.

Table 1. Air craft primer Example.
Numbers in table represent adhesion force

	Application Methods			
Primer Type	Dipping	Spraying	Totals	Average
1	4.0, 4.5, 4.3	5.4, 4.9, 5.6	28.7	4.78
2	5.6, 4.9, 5.4	5.8, 6.1, 6.3	34.1	5.68
3	3.8, 3.7, 4.0	5.5, 5.0, 5.0	27.0	4.50
Totals	40.2	49.6	89.8	
Averages	4.47	5.51		4.99

QUESTIONS OF INTEREST:

Do Primer Type (Factor A) or Application Method (Factor B) have an effect on the adhesion force?

Is it possible that a particular combination of Primer Type and Application Method works better than other combinations?

By Row:

$$y_{i\bullet\bullet} = \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \qquad \overline{y}_{i\bullet\bullet} = \frac{y_{i\bullet\bullet}}{bn}$$

$$(Totals) \qquad (Average)$$

By Column:

$$y_{\bullet j\bullet} = \sum_{i=1}^{a} \sum_{k=1}^{n} y_{ijk} \qquad \overline{y}_{\bullet j\bullet} = \frac{y_{\bullet j\bullet}}{an}$$

$$(Totals) \qquad (Average)$$

By Table:

$$y_{\bullet\bullet\bullet} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \qquad \overline{y}_{\bullet\bullet\bullet} = \frac{y_{\bullet\bullet\bullet}}{a \cdot b \cdot n}$$

$$(Totals) \qquad (Average)$$

	Factor B				
Factor A	1	$\cdots$	b	Totals	Average
1	$y_{111}...y_{11n}$: $\overline{y}_{11\bullet}$	$\cdots$	$y_{1b1}...y_{1bn}$: $\overline{y}_{1b\bullet}$	$y_{1\bullet\bullet}$	$\overline{y}_{1\bullet\bullet}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
a	$y_{a11}...y_{a1n}$: $\overline{y}_{a1\bullet}$	$\cdots$	$y_{ab1}...y_{abn}$: $\overline{y}_{ab\bullet}$	$y_{a\bullet\bullet}$	$\overline{y}_{a\bullet\bullet}$
Totals	$y_{\bullet 1\bullet}$	$\cdots$	$y_{\bullet b\bullet}$	$y_{\bullet\bullet\bullet}$	
Averages	$\overline{y}_{\bullet 1\bullet}$	$\cdots$	$\overline{y}_{\bullet b\bullet}$		$\overline{y}_{\bullet\bullet\bullet}$

$a =$ #Factor A Levels $b =$ # Factor B Levels

$n =$ #Observations within a cell (using a balanced design)

By Cell:

$$y_{ij\bullet} = \sum_{k=1}^{n} y_{ijk} \qquad \overline{y}_{ij\bullet} = \frac{y_{ij\bullet}}{n}$$

$$(Totals) \qquad (Average)$$

Analysis in file "Adhesion_Analysis.R"

Adhesion Force Data

\# A tibble: 18 x 3
\# Groups: Primer, Method [6]

	Force <dbl>	Primer <chr>	Method <chr>
1	4	Type 1	Dipping
2	4.5	Type 1	Dipping
3	4.3	Type 1	Dipping
4	5.4	Type 1	Spraying
5	4.9	Type 1	Spraying
6	5.6	Type 1	Spraying
7	5.6	Type 2	Dipping
8	4.9	Type 2	Dipping
9	5.4	Type 2	Dipping
10	5.8	Type 2	Spraying
11	6.1	Type 2	Spraying
12	6.3	Type 2	Spraying
13	3.8	Type 3	Dipping
14	3.7	Type 3	Dipping
15	4	Type 3	Dipping
16	5.5	Type 3	Spraying
17	5	Type 3	Spraying
18	5	Type 3	Spraying

R-Code to perform analysis by **Primer Type and Method**

```r
table_adhesion_AB<-group_by(adhesion_data, Primer, Method) %>%
  summarise(
    count = n(),
    mean = mean(Force),
    var = var(Force),
    ss = sum((Force-mean(Force))^2)
  )
```

Output of R-Code above:

\# A tibble: 6 x 6
\# Groups: Primer [3]

	Primer <chr>	Method <chr>	count <int>	mean <dbl>	var <dbl>	ss <dbl>
1	Type 1	Dipping	3	4.27	0.0633	0.127
2	Type 1	Spraying	3	5.3	0.130	0.260
3	Type 2	Dipping	3	5.3	0.130	0.260
4	Type 2	Spraying	3	6.07	0.0633	0.127
5	Type 3	Dipping	3	3.83	0.0233	0.0467
6	Type 3	Spraying	3	5.17	0.0833	0.167

```
> mean(table_adhesion_AB$var)
[1] 0.08222222
> sum(table_adhesion_AB$ss)
[1] 0.9866667
```

$SS_E = 0.987$

$df_E = 6*(3-1) = 12$

$MS_E = 0.082 = 0.987/12$

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\overline{y}_{\bullet\bullet\bullet})^2 = \qquad (SS_T)$$

$$bn\sum_{i=1}^{a}(\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet\bullet\bullet})^2 + \qquad (SS_A)$$

$$an\sum_{j=1}^{b}(\overline{y}_{\bullet j\bullet}-\overline{y}_{\bullet\bullet\bullet})^2 + \qquad (SS_B)$$

$$n\sum_{i=1}^{a}\sum_{j=1}^{b}(\overline{y}_{ij\bullet}-\overline{y}_{i\bullet\bullet}-\overline{y}_{\bullet j\bullet}+\overline{y}_{\bullet\bullet\bullet})^2 + \qquad (SS_{AB})$$

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\overline{y}_{ij\bullet})^2 \qquad (SS_E)$$

or

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

The corresponding degrees of freedom decomposition is

$$abn-1=(a-1)+(b-1)+(a-1)(b-1)+ab(n-1)$$

Analysis in file "Adhesion_Analysis.R"

Adhesion Force Data

```
# A tibble: 18 x 3
# Groups:   Primer, Method [6]
   Force Primer Method
   <dbl> <chr>  <chr>
 1   4   Type 1 Dipping
 2   4.5 Type 1 Dipping
 3   4.3 Type 1 Dipping
 4   5.4 Type 1 Spraying
 5   4.9 Type 1 Spraying
 6   5.6 Type 1 Spraying
 7   5.6 Type 2 Dipping
 8   4.9 Type 2 Dipping
 9   5.4 Type 2 Dipping
10   5.8 Type 2 Spraying
11   6.1 Type 2 Spraying
12   6.3 Type 2 Spraying
13   3.8 Type 3 Dipping
14   3.7 Type 3 Dipping
15   4   Type 3 Dipping
16   5.5 Type 3 Spraying
17   5   Type 3 Spraying
18   5   Type 3 Spraying
```

```
> mu<-mean(adhesion_data$Force)
> mu
[1] 4.988889
> SS_T<-sum((adhesion_data$Force-mu)^2)
> SS_T
[1] 10.71778
```

R-Code to perform analysis by **Primer Type:**

```
table_adhesion_A<-group_by(adhesion_data, Primer) %>%
  summarise(
    count = n(),
    mean = mean(Force)
  )
```

Output of R-Code above:

```
# A tibble: 3 x 3
  Primer count  mean
  <chr>  <int> <dbl>
1 Type 1     6  4.78
2 Type 2     6  5.68
3 Type 3     6  4.5
```

n = 3, Number of Cell Observatoon

a = 3, Number of Factor levels A

df_a = 3-1 = 2, b * n = 6

```
> SS_A<-sum(table_adhesion_A$count*(table_adhesion_A$mean-mu)^2)
> SS_A
[1] 4.581111
```

Analysis in file "Adhesion_Analysis.R"

Adhesion Force Data

```
# A tibble: 18 x 3
# Groups:   Primer, Method [6]
   Force Primer Method
   <dbl> <chr>  <chr>
 1   4   Type 1 Dipping
 2   4.5 Type 1 Dipping
 3   4.3 Type 1 Dipping
 4   5.4 Type 1 Spraying
 5   4.9 Type 1 Spraying
 6   5.6 Type 1 Spraying
 7   5.6 Type 2 Dipping
 8   4.9 Type 2 Dipping
 9   5.4 Type 2 Dipping
10   5.8 Type 2 Spraying
11   6.1 Type 2 Spraying
12   6.3 Type 2 Spraying
13   3.8 Type 3 Dipping
14   3.7 Type 3 Dipping
15   4   Type 3 Dipping
16   5.5 Type 3 Spraying
17   5   Type 3 Spraying
18   5   Type 3 Spraying
```

```
> mu<-mean(adhesion_data$Force)
> mu
[1] 4.988889
> SS_T<-sum((adhesion_data$Force-mu)^2)
> SS_T
[1] 10.71778
```

R-Code to perform analysis by **Method:**

```
table_adhesion_B<-group_by(adhesion_data, Method) %>%
  summarise(
    count = n(),
    mean = mean(Force)
  )
```

Output of R-Code above:

```
# A tibble: 2 x 3
  Method   count  mean
  <chr>    <int> <dbl>
1 Dipping      9  4.47
2 Spraying     9  5.51
```

n = 3, Number of Cell Observatoon

b = 2, Number of Factor levels A

$df_b = 2\text{-}1 = 1$, $a^*n = 9$

```
> SS_B<-sum(table_adhesion_B$count*(table_adhesion_B$mean-mu)^2)
> SS_B
[1] 4.908889
```

```
> SS_AB<-SS_T-SS_A-SS_B-SS_E
> SS_AB
[1] 0.2411111
```

$$MS_A = \frac{SS_A}{a-1},\ E[MS_A] = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1}$$

$$MS_B = \frac{SS_B}{b-1}, E[MS_B] = \sigma^2 + \frac{bn\sum_{j=1}^{b}\beta_i^2}{b-1}$$

$$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}, E[MS_{AB}] = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

$$MS_E = \frac{SS_E}{ab(n-1)}, E[MS_E] = \sigma^2$$

Model Description:

$$Y_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}, \begin{cases} i = 1, \ldots, a \\ j = 1, \ldots, b \\ k = 1, \ldots, n \end{cases}$$

μ	: a parameter common to all treatments called *the overall mean*
α_i	: a parameter unique to the i-th level of Factor A,
β_j	: a parameter unique to the j-th level of Factor B,
$(\alpha\beta)_{ij}$	: a parameter unique to the i-th level of Factor A and j-th level of Factor B, it models a potential interaction effect
ϵ_{ijk}	: a random error component, $\epsilon_{ijk} \sim N(0, \sigma)$ for all i, j, k and $i.i.d.$

Hypothesis Tests of interest:

1. $H_0 : \alpha_1 = \alpha_2 = \ldots = \alpha_a = 0,\ H_1$: at least one $\alpha_i \neq 0$
2. $H_0 : \beta_1 = \beta_2 = \ldots = \beta_b = 0,\ H_1$: at least one $\beta_j \neq 0$
3. $H_0 : (\alpha\beta)_{ij} = 0$ for all $i, j,\ H_1$: at least one$(\alpha\beta)_{ij} \neq 0$

- Assuming that the null-hypotheses above are true it can be shown that:

$$\frac{MS_A}{MS_E} \sim F_{a-1,ab(n-1)}, \frac{MS_B}{MS_E} \sim F_{b-1,ab(n-1)}, \frac{MS_{AB}}{MS_E} \sim F_{(a-1)(b-1),ab(n-1)}$$

ANALYSIS OF VARIANCE (ANOVA) TABLE:

General Format:

Source	SS	df	MS	F
Factor A	SS_A	$a-1$	$\frac{SS_A}{a-1}$	$\frac{MS_A}{MS_E}$
Factor B	SS_B	$b-1$	$\frac{SS_B}{b-1}$	$\frac{MS_B}{MS_E}$
AB Interaction	SS_{AB}	$(a-1)(b-1)$	$\frac{SS_{AB}}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$\frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

Aircraft Primer Paint Example:

SOURCE	SS	df	MS	F-value	p-value
A	4.58	2	2.29	27.86	0.00 %
B	4.91	1	4.91	59.70	0.00 %
AB	0.24	2	0.12	1.47	26.93 %
Error	0.99	12	0.08		
Total	10.72	17			

Conclusion:

- The interaction between Primer Type (A) and Application Method (B) does not add to variability (which is good).
- Application method (B) appears to contribute more to variability in adhesion force than primer type (A).
- Based on the average adhesion force the best combination seems to choose Primer "Type 2" and the "Spraying" Method since there is "no interaction".

Recall: $$Y_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \epsilon_{ijk}, \begin{cases} i = 1, \ldots, a \\ j = 1, \ldots, b \\ k = 1, \ldots, n \end{cases}$$

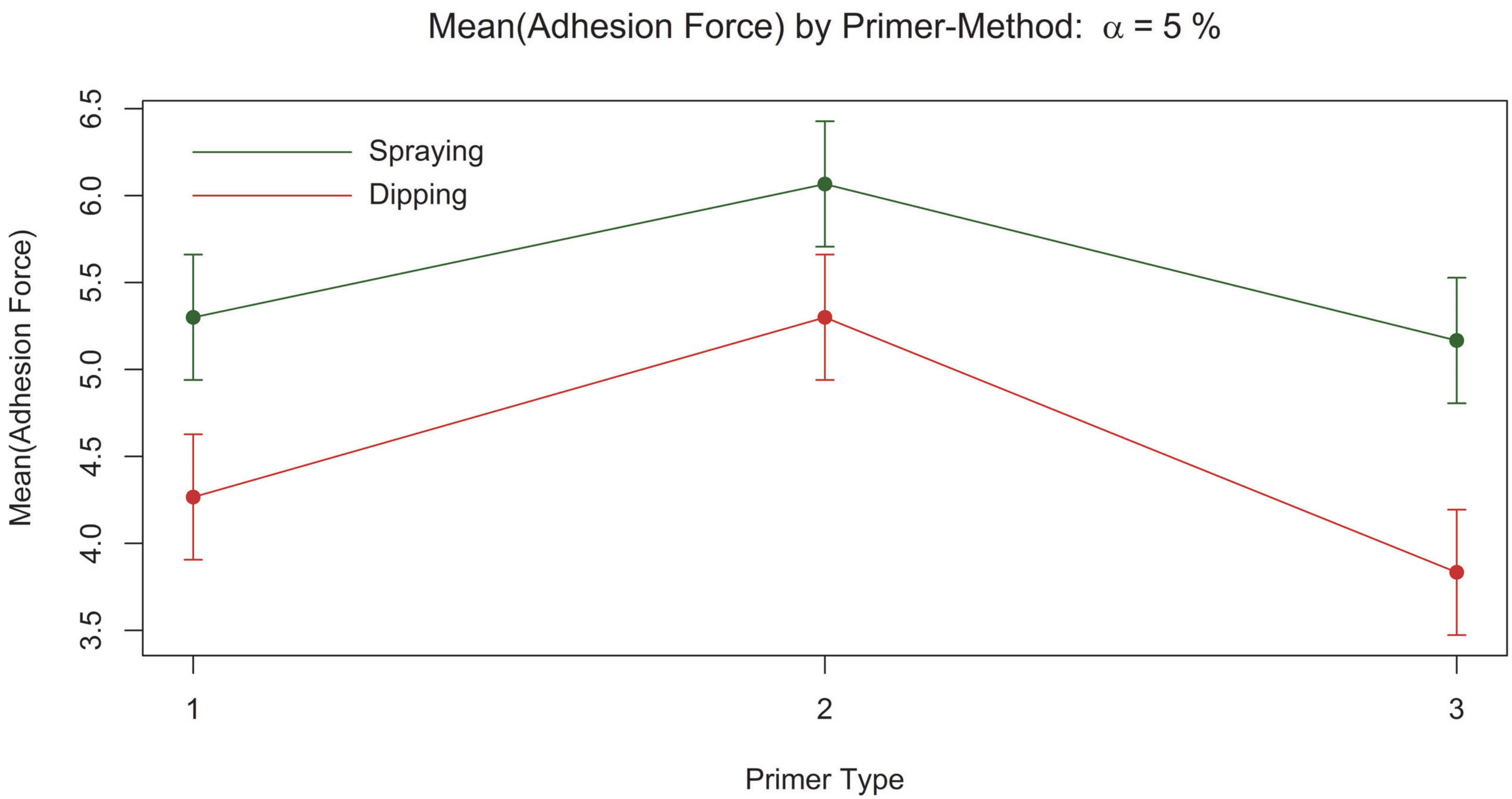
ϵ_{ijk} : a random error component, $\epsilon_{ijk} \sim N(0, \sigma)$ for all i, j, k and

$$\begin{aligned} &\widehat{\mu} = \overline{y}_{\bullet\bullet\bullet} \\ &\widehat{\alpha}_i = \overline{y}_{i\bullet\bullet} - \overline{y}_{\bullet\bullet\bullet}, \widehat{\beta}_j = \overline{y}_{\bullet j\bullet} - \overline{y}_{\bullet\bullet\bullet}, \\ &(\widehat{\alpha\beta})_{ij} = \overline{y}_{ij\bullet} - (\overline{y}_{\bullet\bullet\bullet} + \widehat{\alpha}_i + \widehat{\beta}_j) \\ &\widehat{\sigma}^2 = MS_E = SS_E/[ab(n-1)] \\ &\widehat{\mu}_{ij} = \overline{y}_{ij\bullet} \end{aligned}$$

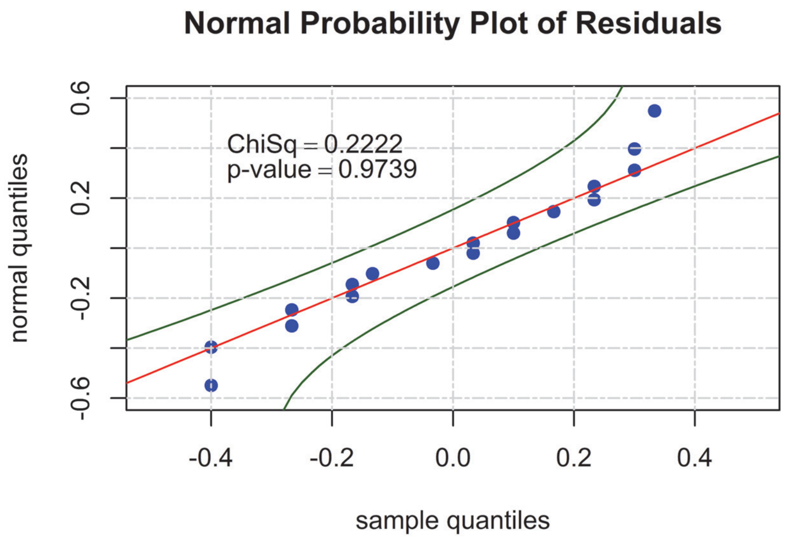
$100(1-\alpha)\%$ confidence intervals means μ_i, μ_j, μ_{ij}:

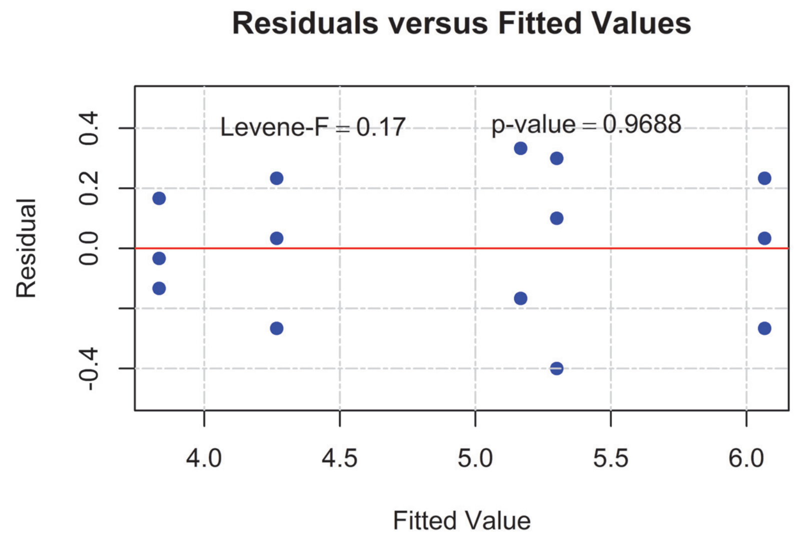
$$\overline{y}_{i\bullet\bullet} \pm t_{\alpha/2, ab(n-1)}\sqrt{\frac{MS_E}{bn}}, \overline{y}_{\bullet j\bullet} \pm t_{\alpha/2, ab(n-1)}\sqrt{\frac{MS_E}{an}}, \overline{y}_{ij\bullet} \pm t_{\alpha/2, ab(n-1)}\sqrt{\frac{MS_E}{n}}$$

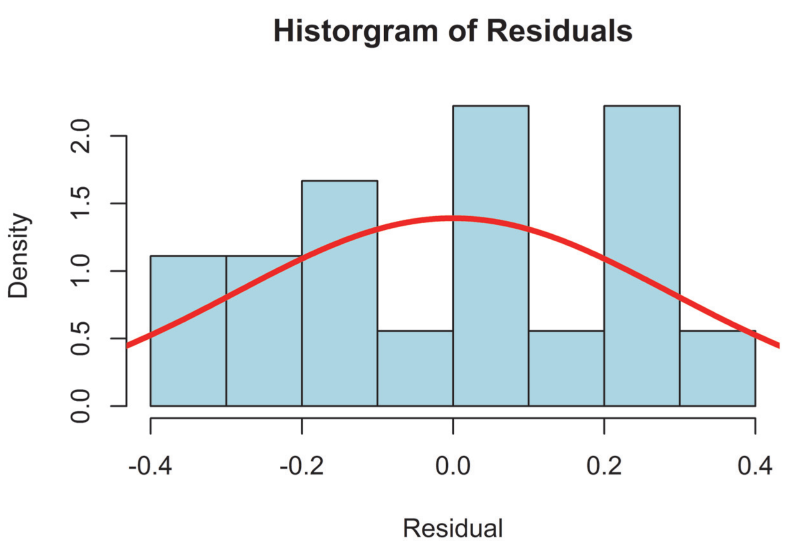
Analysis in file "Adhesion_Analysis.R"

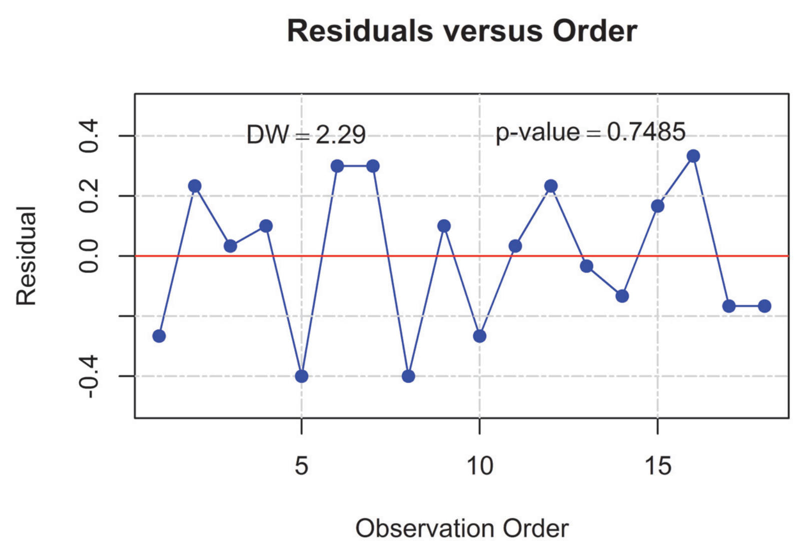


Conclusion: Choose **Primer Type 2** and the **Spaying Method**. Confidence Intervals do not overlap (and one observes there is no interaction).


Normal Probability Plot of Residuals
ChiSq = 0.2222
p-value = 0.9739
normal quantiles
sample quantiles
-0.6
-0.2
0.2
0.6
-0.4
-0.2
0.0
0.2
0.4


Residuals versus Fitted Values
Levene-F = 0.17
p-value = 0.9688
Residual
Fitted Value
-0.4
0.0
0.2
0.4
4.0
4.5
5.0
5.5
6.0


Historgram of Residuals
Density
Residual
0.0
0.5
1.0
1.5
2.0
-0.4
-0.2
0.0
0.2
0.4


Residuals versus Order
DW = 2.29
p-value = 0.7485
Residual
Observation Order
-0.4
0.0
0.2
0.4
5
10
15

```
alpha<-table_adhesion_A$mean-mu
table_adhesion_A<-bind_cols(table_adhesion_A,"alpha"=alpha)
```

```
beta<-table_adhesion_B$mean-mu
table_adhesion_B<-bind_cols(table_adhesion_B,"beta"=beta)
```

```
# A tibble: 3 x 6
  Primer count  mean  alpha Conf.LB Conf.UB
  <chr>  <int> <dbl>  <dbl>   <dbl>   <dbl>
1 Type 1     6  4.78 -0.206    4.53    5.04
2 Type 2     6  5.68  0.694    5.43    5.94
3 Type 3     6  4.5  -0.489    4.24    4.76
```

```
# A tibble: 2 x 6
  Method   count  mean   beta Conf.LB Conf.UB
  <chr>    <int> <dbl>  <dbl>   <dbl>   <dbl>
1 Dipping      9  4.47 -0.522    4.26    4.67
2 Spraying     9  5.51  0.522    5.30    5.72
```

```
alpha_beta<-replicate(a*b,0)
for (i in c(1:(a*b))) {
  alpha_A<-table_adhesion_A %>%
             filter(Primer==table_adhesion_AB$Primer[i])
  beta_B<-table_adhesion_B %>%
             filter(Method==table_adhesion_AB$Method[i])
  alpha_beta[i]<-table_adhesion_AB$mean[i]-(mu+alpha_A$alpha+beta_B$beta)
}
table_adhesion_AB<-bind_cols(table_adhesion_AB,"alpha_beta"=alpha_beta)
```

```
# A tibble: 6 x 9
# Groups:   Primer [3]
  Primer Method   count  mean    var     ss alpha_beta Conf.LB Conf.UB
  <chr>  <chr>    <int> <dbl>  <dbl>  <dbl>      <dbl>   <dbl>   <dbl>
1 Type 1 Dipping      3  4.27 0.0633 0.127     0.00556    3.91    4.63
2 Type 1 Spraying     3  5.3  0.130  0.260    -0.00556    4.94    5.66
3 Type 2 Dipping      3  5.3  0.130  0.260     0.139      4.94    5.66
4 Type 2 Spraying     3  6.07 0.0633 0.127    -0.139      5.71    6.43
5 Type 3 Dipping      3  3.83 0.0233 0.0467   -0.144      3.47    4.19
6 Type 3 Spraying     3  5.17 0.0833 0.167     0.144      4.81    5.53
```

ANOVA: Force versus Primer, Method

Factor Information

Factor	Type	Levels	Values
Primer	Fixed	3	Type 1, Type 2, Type 3
Method	Fixed	2	Dipping, Spraying

Analysis of Variance for Force

Source	DF	SS	MS	F	P
Primer	2	4.5811	2.29056	27.86	0.000
Method	1	4.9089	4.90889	59.70	0.000
Primer*Method	2	0.2411	0.12056	1.47	0.269
Error	12	0.9867	0.08222		
Total	17	10.7178			

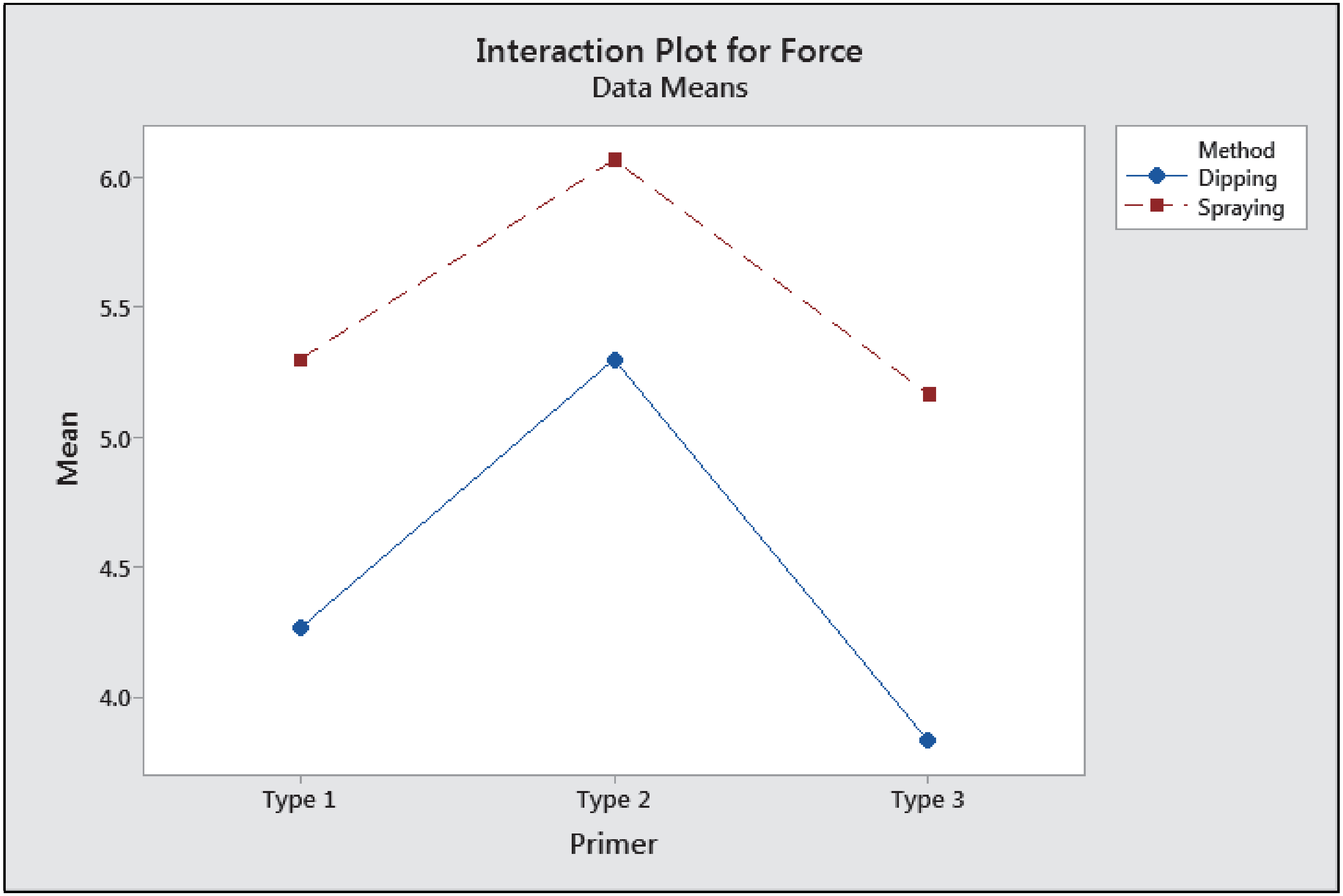
Model Summary

S	R-sq	R-sq(adj)
0.286744	90.79%	86.96%

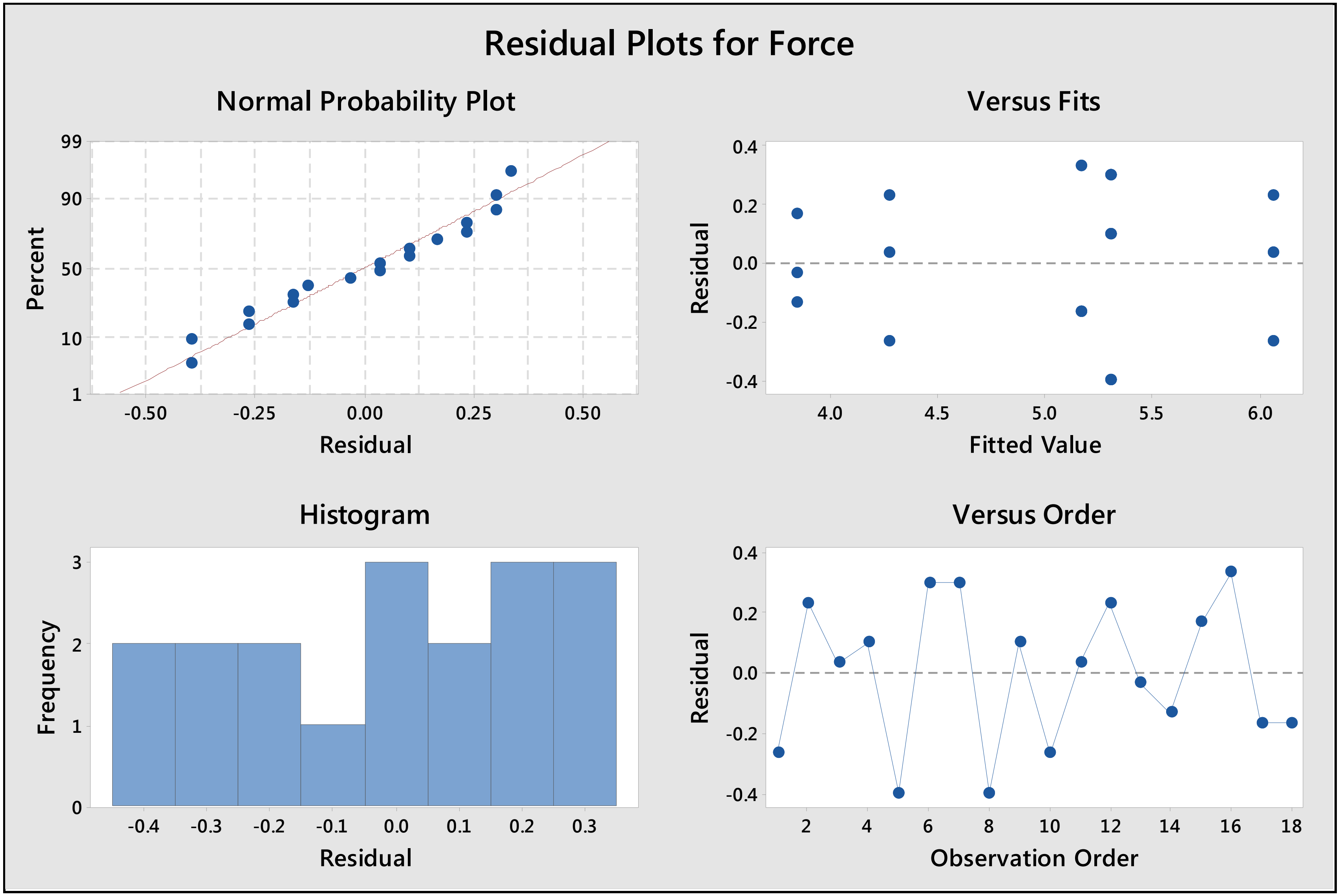
R-Code

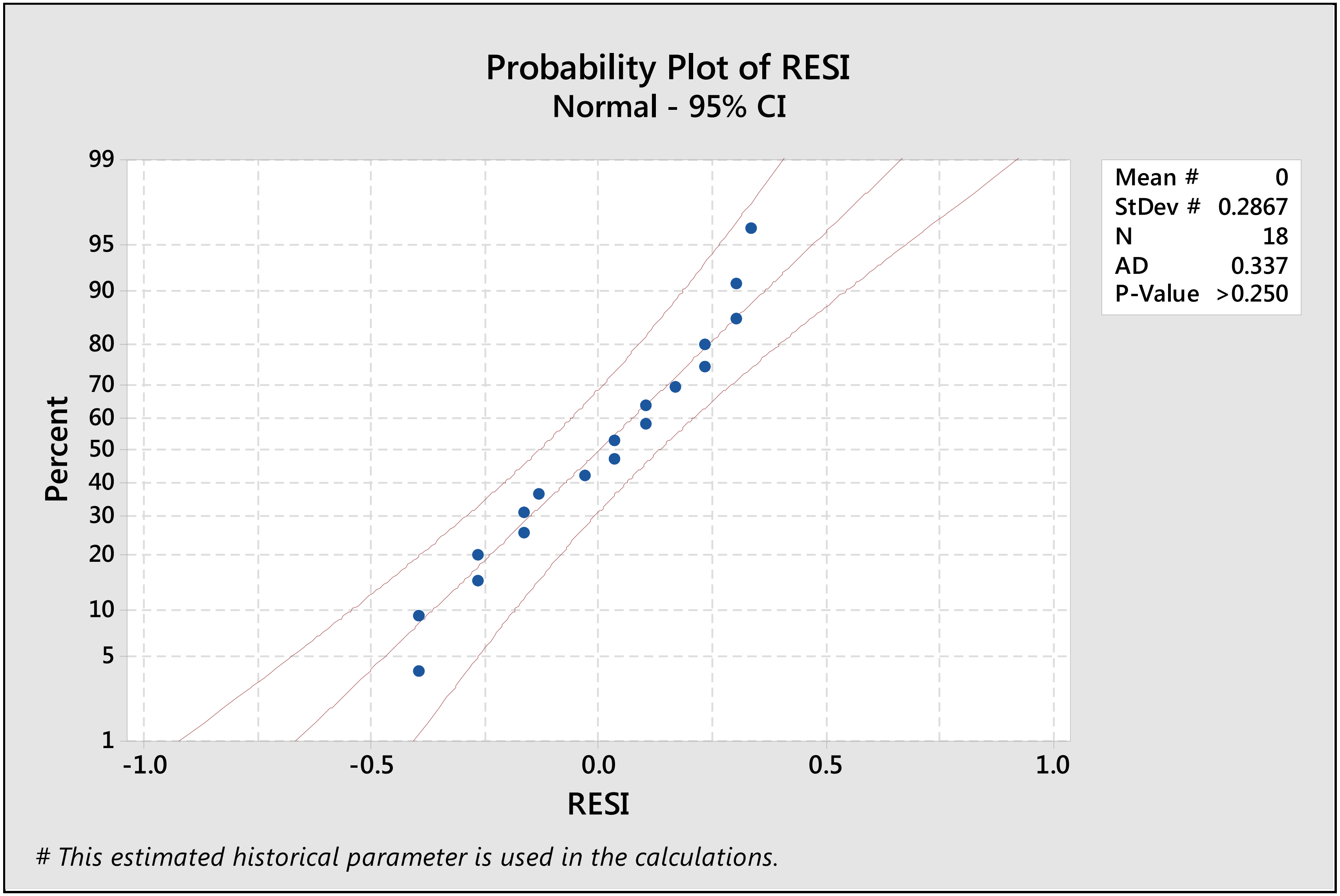
```
> R_squared<-(1-SS_E/SS_T)
> R_squared
[1] 0.9079411
```

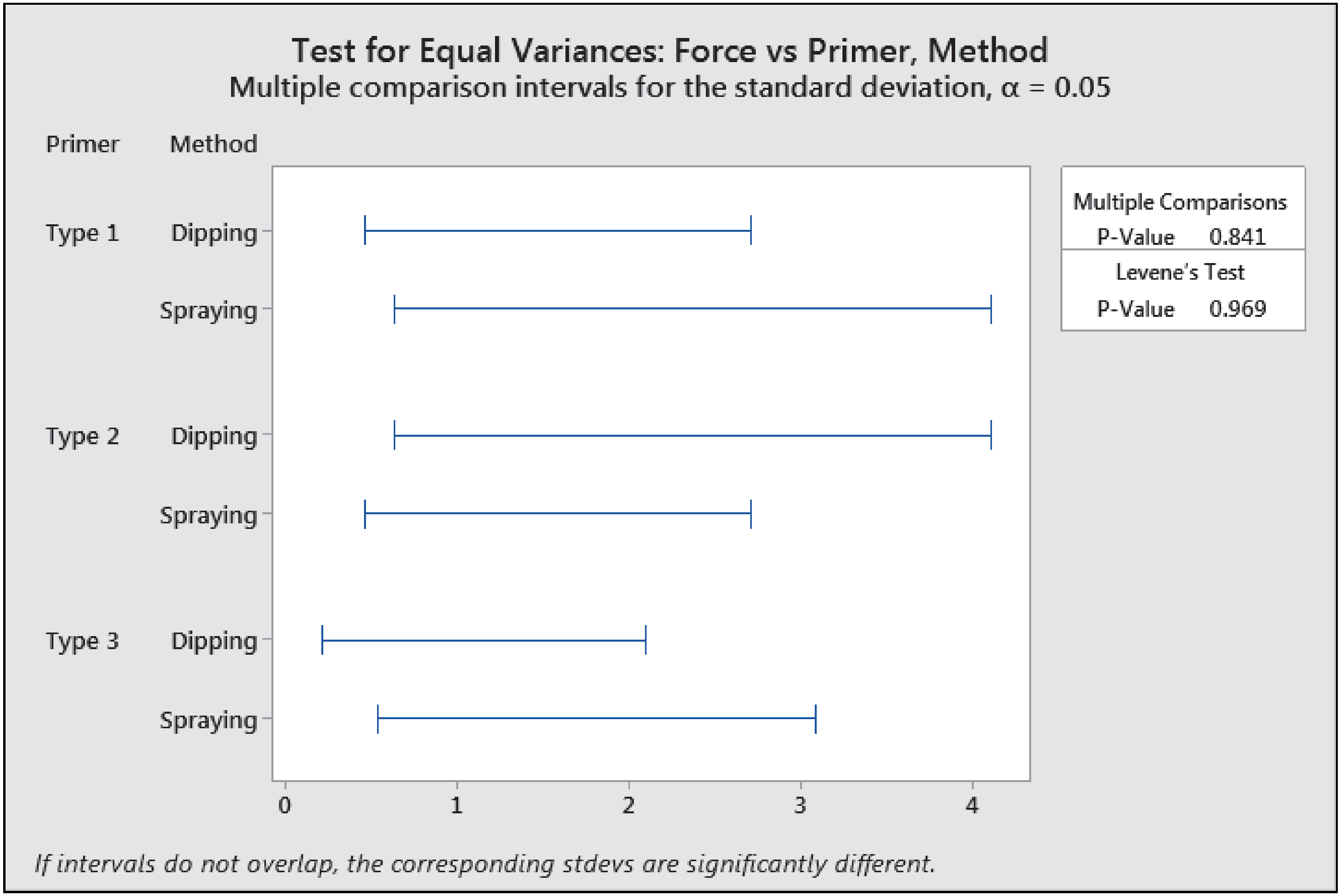
```
> R_squared_adj<-(1-MS_E/var_force)
> R_squared_adj
[1] 0.8695832
```



No Confidence Intervals are plotted in Minitab using MS_E


Residual Plots for Force
Normal Probability Plot
Percent
99
90
50
10
1
-0.50
-0.25
0.00
0.25
0.50
Residual
Versus Fits
Residual
0.4
0.2
0.0
-0.2
-0.4
4.0
4.5
5.0
5.5
6.0
Fitted Value
Histogram
Frequency
3
2
1
0
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
Residual
Versus Order
Residual
0.4
0.2
0.0
-0.2
-0.4
2
4
6
8
10
12
14
16
18
Observation Order


Probability Plot of RESI
Normal - 95% CI
Percent
99
95
90
80
70
60
50
40
30
20
10
5
1
-1.0
-0.5
0.0
0.5
1.0
RESI
Mean # 0
StDev # 0.2867
N 18
AD 0.337
P-Value >0.250
This estimated historical parameter is used in the calculations.


Test for Equal Variances: Force vs Primer, Method
Multiple comparison intervals for the standard deviation, α = 0.05
Primer
Method
Type 1
Dipping
Spraying
Type 2
Dipping
Spraying
Type 3
Dipping
Spraying
0
1
2
3
4
Multiple Comparisons
P-Value 0.841
Levene's Test
P-Value 0.969
If intervals do not overlap, the corresponding stdevs are significantly different.

2^K Factorial Design: K Factors each factor at two levels
(High, Low) = ($+$, $-$)

- Special Experiment or Run Notation:

  (i) Use lowercase letter of factor if high level of factor is present.

  (ii) Do not include letter of factor if low level of factor is present.

  (iii) Use (1) to indicate all factors at low level

Example 3 factors A, B, C: $\Rightarrow$ ac indicates $+$ for A, $-$ for B, $+$ for C
$\Rightarrow$ (1) indicates $-$ for A, $-$ for B, $-$ for C

- For analysis purposes, let these letters (e.g. (1), a, b, ab) also denote the totals (sums) of all observations in these experiments (or runs)

2^2 Factorial Design: The Router Experiment
(Hence, this is the same as a TWO-WAY ANOVA with 2 levels per factor)

A router is used to cut registration notches in printed circuit boards. The average notch dimension is satisfactory and the process is in statistical control, but there is too much variability in the process. This excess variability leads to problems in board assembly. The components are inserted into the board using automatic equipment, and the variability in notch dimension causes improper board registration. A a result, the auto-insertion equipment does not work properly.

Since the process is in Statistical Control, the Quality Improvement team assigned to this project decided to use a designed experiment to study the process. The team considered two factors: bit size (A) and speed (B). Two levels were chosen for each factor (bit size A at $\frac{1}{16}''$ and $\frac{1}{8}''$ and drill speed B at $40 rpm$ and $80\, rpm$) and a 2^2 design was set up. For each run (experiment setup) four different test were conducted (four replications).

Table 2: Router Experiment Data ($n = 4$)

Run		A	B	$Vibration$				$Totals$
1	(1)	$-$	$-$	18.2	18.9	12.9	14.4	64.4
2	a	$+$	$-$	27.2	24.0	22.4	22.5	96.1
3	b	$-$	$+$	15.9	14.5	15.1	14.2	59.7
4	ab	$+$	$+$	41.0	43.9	36.3	39.9	161.7

Example: Two Factors, n Replications

		FACTOR B			
		$-$	$+$	TOTALS	AVERAGE
FACTOR	$-$	(1)	b	$(1)+b$	$\frac{(1)+b}{2n}$
A	$+$	a	ab	$a+ab$	$\frac{a+ab}{2n}$
TOTALS		$(1)+a$	$b+ab$		
AVERAGE		$\frac{(1)+a}{2n}$	$\frac{b+ab}{2n}$		

Effect of A: **A difference of two mean values**

$$\frac{a+ab}{2n} - \frac{(1)+b}{2n} = \frac{1}{2n}[a+ab-b-(1)]$$

Effect of B: **A difference of two mean values**

$$\frac{b+ab}{2n} - \frac{(1)+a}{2n} = \frac{1}{2n}[b\ +ab-a-(1)]$$

AB Interaction Effect: **A difference of two mean values**

$$\frac{ab-a}{2n} - \frac{b-(1)}{2n} = \frac{1}{2n}[ab+(1)-a-b]$$

(Difference in effect of B while keeping Factor A the same)

- The elements in brackets are contrasts with a single degree of freedom (Recall from One-Way Anova!)

- Contrast coefficients have $+$ or $-$ sign, that can be summarized in a table.

CONTRAST TABLE FOR 2^2

		FACTOR		
RUN	TREATMENT	A	B	AB
1	(1)	$-$	$-$	$+$
2	a	$+$	$-$	$-$
3	b	$-$	$+$	$-$
4	ab	$+$	$+$	$+$

Definition Contrast A, B and AB: **Observe AB collumn $= A \times B$ collumns**

$$C_A = [-(1) + a - b + ab] = [a + ab - b - (1)]$$

$$C_B = [-(1) - a + b + ab] = [b + ab - a - (1)]$$

$$C_{AB} = [(1) - a - b + ab] = [(1) + ab - a - b]$$

- Note that these three contrasts are orthogonal. Hence, hypothesis tests involving them are independent (Recall the ONE-WAY ANOVA notes).

Effects of A, B and AB (Recall $(K = 2)$) :

$$Effect_A = \frac{C_A}{2^{K-1}n}; Effect_A = \frac{C_B}{2^{K-1}n}; Effect_{AB} = \frac{C_{AB}}{2^{K-1}n}$$

Sums of Squares of A, B and AB:

$$SS_A = \frac{C_A^2}{2^K n};\ SS_B = \frac{C_B^2}{2^K n};\ SS_{AB} = \frac{C_{AB}^2}{2^K n};$$

$$SS_T = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}(y_{ijk} - \overline{y}_{\bullet\bullet\bullet})^2 = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\bullet\bullet\bullet}^2}{4n},$$

$$y_{\bullet\bullet\bullet} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} y_{ijk}$$

$$SS_E = SS_T - SS_A - SS_B - SS_{AB}$$

GENERAL FORMAT OF A 2^2 ANOVA TABLE

Source	SS	df	MS	F
Factor A	$\frac{[a+ab-b-(1)]^2}{4n}$	1	SS_A	$\frac{MS_A}{MS_E}$
Factor B	$\frac{[b+ab-a-(1)]^2}{4n}$	1	SS_B	$\frac{MS_B}{MS_E}$
AB Interaction	$\frac{[ab+(1)-a-b]^2}{4n}$	1	SS_{AB}	$\frac{MS_{AB}}{MS_E}$
Error	$SS_T - SS_A - SS_B - SS_{AB}$	$4(n-1)$	$\frac{SS_E}{4(n-1)}$	
Total	$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\dots}^2}{4n}$	$4n-1$		

		FACTOR B			
		Drill Speed			
		$-$	$+$	TOTALS	AVERAGE
Factor A	$-$	64.4	59.7	124.1	15.5
Bit Size	$+$	96.1	161.1	257.2	32.2
TOTALS		160.5	220.8		
AVERAGE		20.1	27.6		

Definition Contrast A, B and AB:

$$C_A = [-64.4 + 96.1 - 59.7 + 161.1] = 133.10,$$
$$C_B = [-64.4 - 96.1 + 59.7 + 161.1] = 60.3,$$
$$C_{AB} = [64.4 - 96.1 - 59.7 + 161.1] = 69.7$$

$$Effect_A = \frac{133.10}{2 \times 4} = 16.6, Effect_B = \frac{60.3}{2 \times 4} = 7.5, Effect_{AB} = \frac{69.7}{2 \times 4} = 8.7$$

Sums of Squares of A, B and AB:

$$S_A = \frac{(133.10)^2}{2^2 \times 4} = 1107.2;\; SS_B = \frac{(60.3)^2}{2^2 \times 4} = 227.26\,;\; SS_{AB} = \frac{(69.7)^2}{2^2 \times 4} = 303.63$$

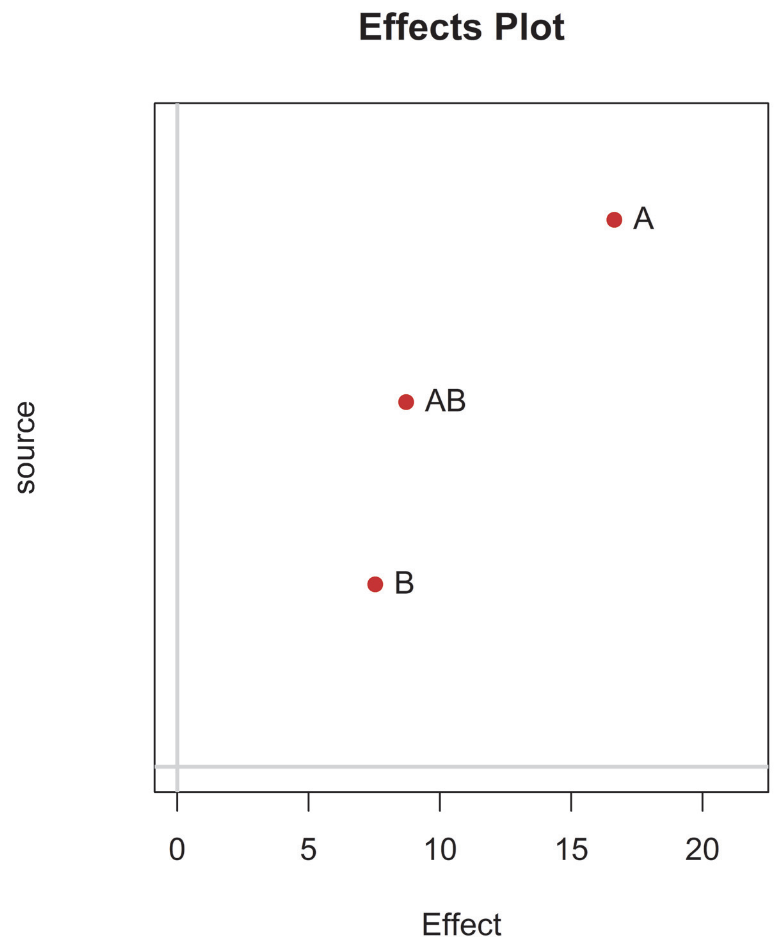
$$SS_T = 1709.83, SS_E = 1709.83 - 1107.2 - 227.26 - 303.63 = 71.72$$

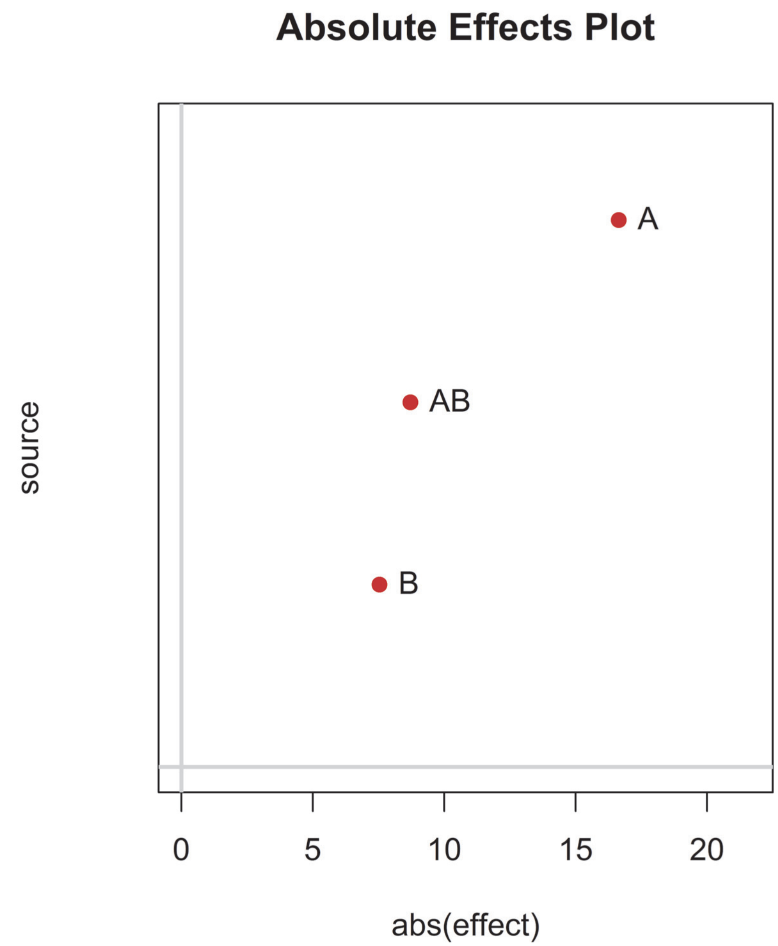
ANOVA TABLE ROUTER EXPERIMENT

Source	SS	df	MS	F	P-Value
Bit Size (Factor A)	1107.2	1	1107.2	185.25	$1.2\,e-8$
Drill Speed (Factor B)	227.26	1	227.26	38.02	$4.8\,e-5$
AB Interaction	303.63	1	303.63	50.8	$1.2\,e-5$
Error	71.72	12	5.97		
Total	1709.83	15			

CONCLUSION: Bit Size (A) and Higher Drill Speed (B) contribute to increased vibration. However, due to the interaction it is more advantageous to use a small drill bit (A) and run it a high speed (B), rather than running it at a low speed (B).

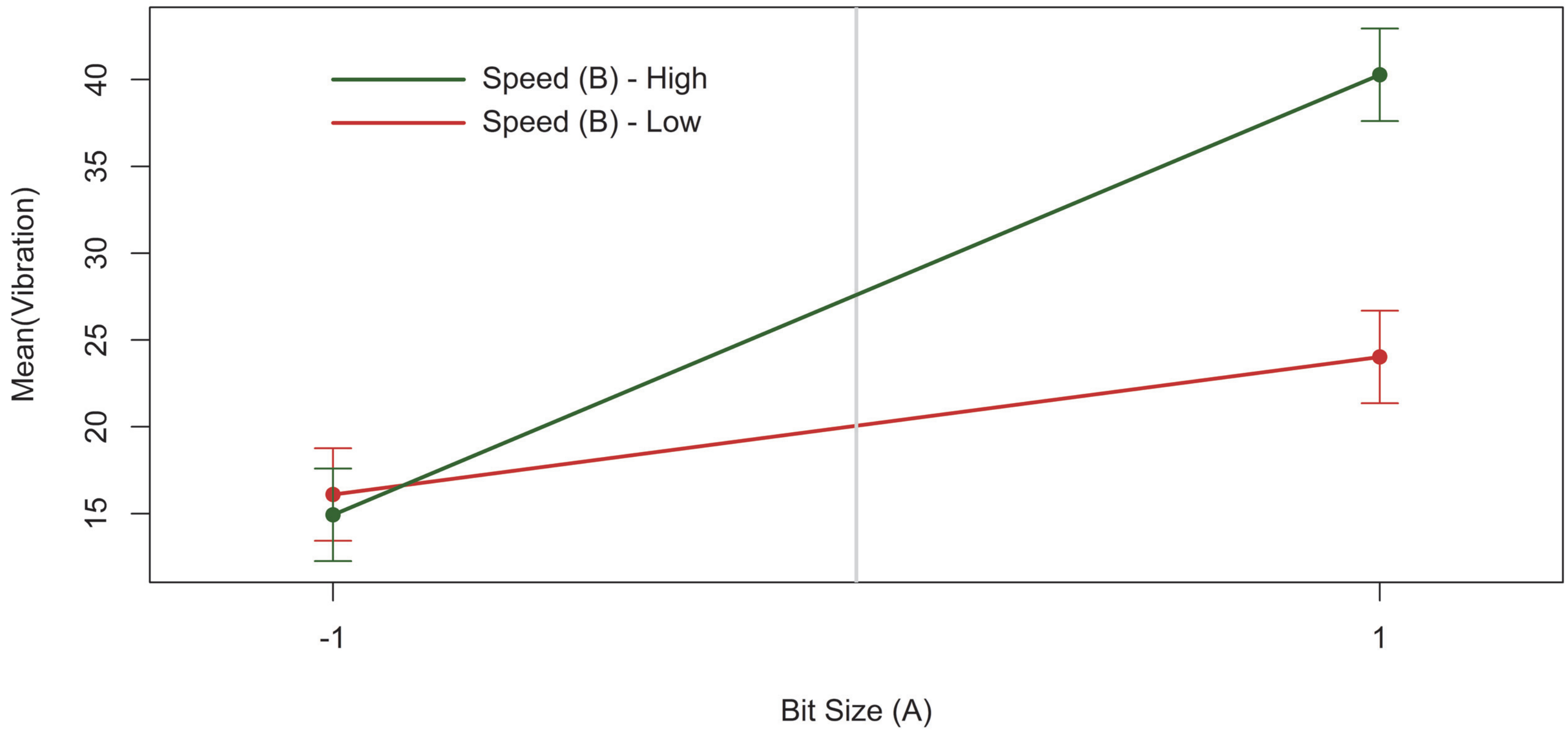
Analysis in file "Routing_Experiment.R"





Analysis in file "Routing_Experiment.R"

Interaction Plot BIT SIZE (A) versus DRILL SPEED (B): α = 5 %



Conclusion: Choose a **Small bit size (A)** and **High drill speed (B)**, **but confidence intervals overlap** (and one observes the interaction)**!**

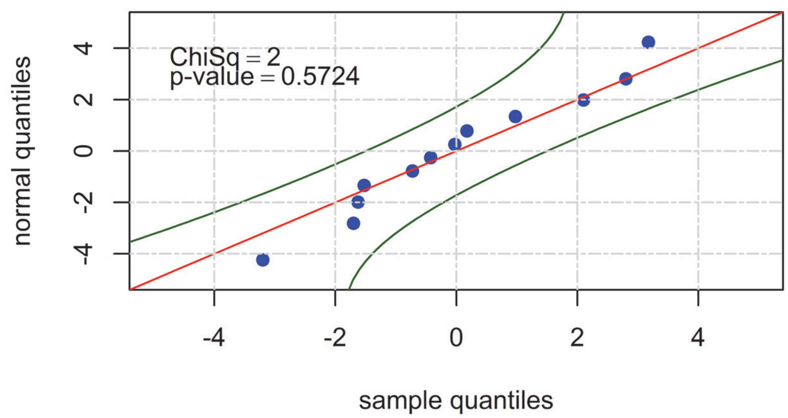
Analysis in file "Routing_Experiment.R"

```
# A tibble: 4 x 6
  Runs    `1`   `2`   `3`   `4`   sum
  <chr> <dbl> <dbl> <dbl> <dbl> <dbl>
1 (1)    18.2  18.9  12.9  14.4  64.4
2 a      27.2  24    22.4  22.5  96.1
3 b      15.9  14.5  15.1  14.2  59.7
4 ab     41    43.9  36.3  39.9 161.
```

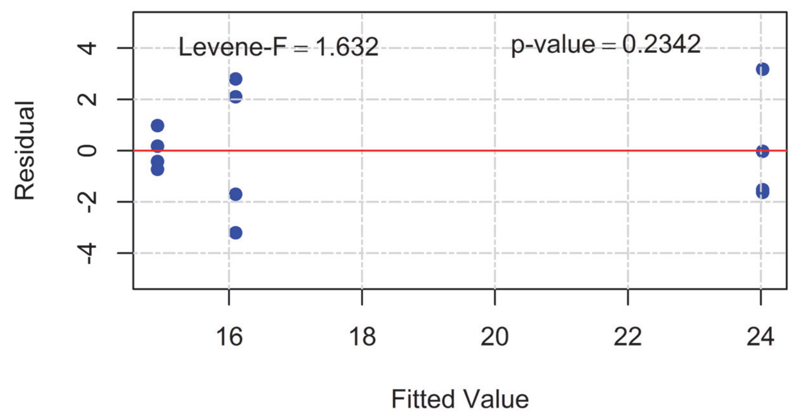
```
   on_off contrasts        SS effects   coeff
A       1     133.1 1107.2256 16.6375 8.31875
B       1      60.3  227.2556  7.5375 3.76875
AB      1      69.7  303.6306  8.7125 4.35625
```

SOURCE	SS	df	MS	F-value	p-value
A	1107.23	1	1107.23	185.25	0.00 %
B	227.26	1	227.26	38.02	0.00 %
AB	303.63	1	303.63	50.80	0.00 %
Error	71.72	12	5.98		
Total	1709.83	15			

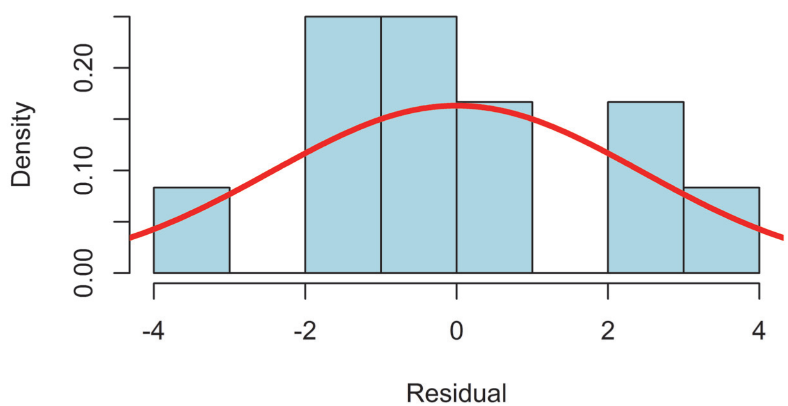
Normal Probability Plot of Residuals



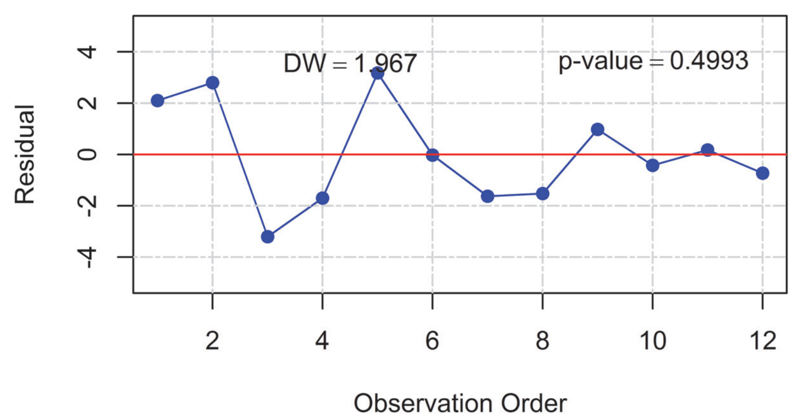
Residuals versus Fitted Values



Historgram of Residuals



Residuals versus Order



Run	Bit Size	Drill Speed	Vibration
(1)	Small	Slow	18.2
(1)	Small	Slow	18.9
(1)	Small	Slow	12.9
(1)	Small	Slow	14.4
a	Big	Slow	27.2
a	Big	Slow	24
a	Big	Slow	22.4
a	Big	Slow	22.5
b	Small	High	15.9
b	Small	High	14.5
b	Small	High	15.1
b	Small	High	14.2
ab	Big	High	41
ab	Big	High	43.9
ab	Big	High	36.3
ab	Big	High	39.9

ANOVA: Vibration versus Bit Size, Drill Speed

Factor Information

Factor	Type	Levels	Values
Bit Size	Fixed	2	Big, Small
Drill Speed	Fixed	2	High, Slow

Analysis of Variance for Vibration

Source	DF	SS	MS	F	P
Bit Size	1	1107.23	1107.23	185.25	0.000
Drill Speed	1	227.26	227.26	38.02	0.000
Bit Size*Drill Speed	1	303.63	303.63	50.80	0.000
Error	12	71.72	5.98		
Total	15	1709.83			

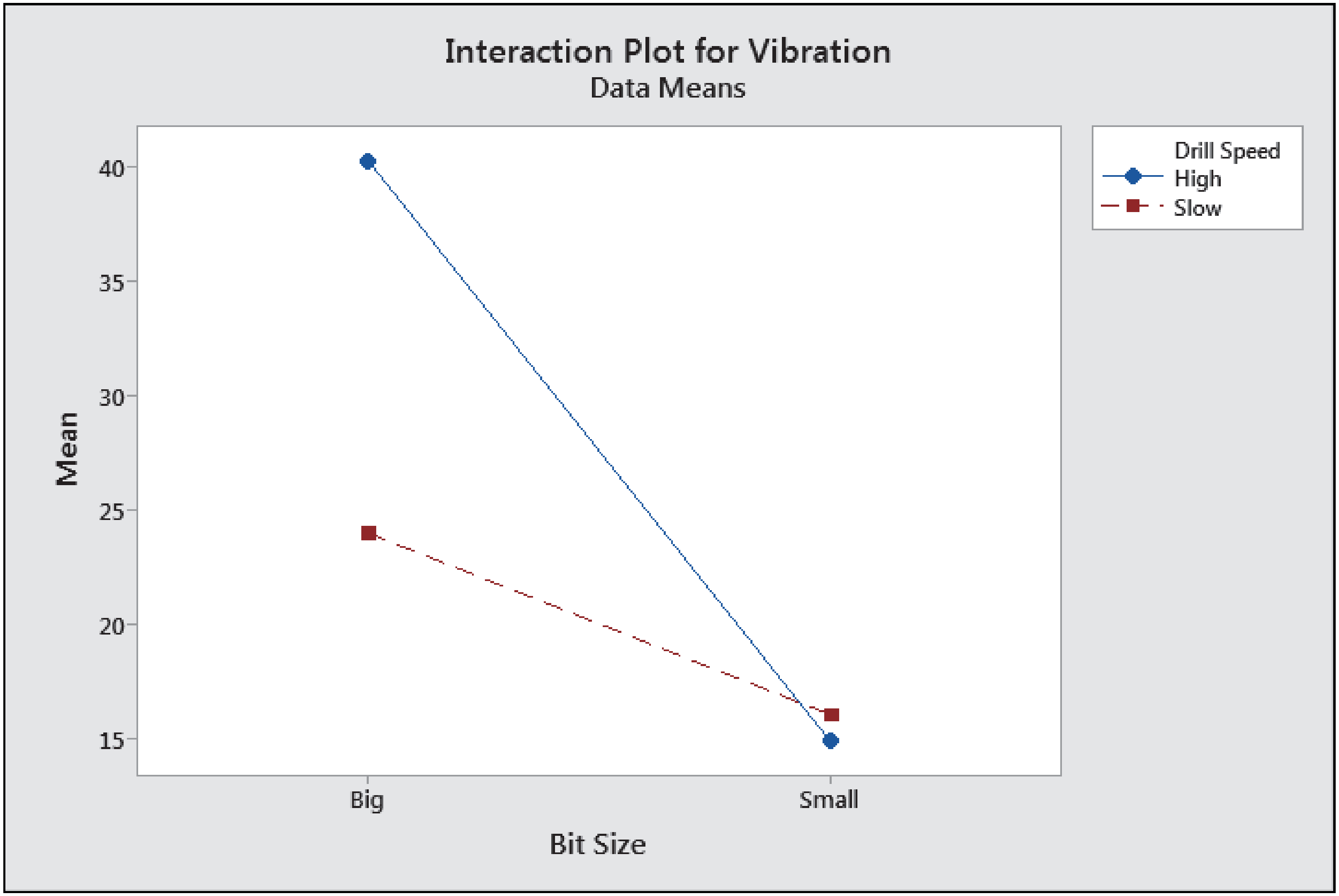
```
> R_squared<-(1-SS_E/SS_T)
> R_squared
[1] 0.958053
```

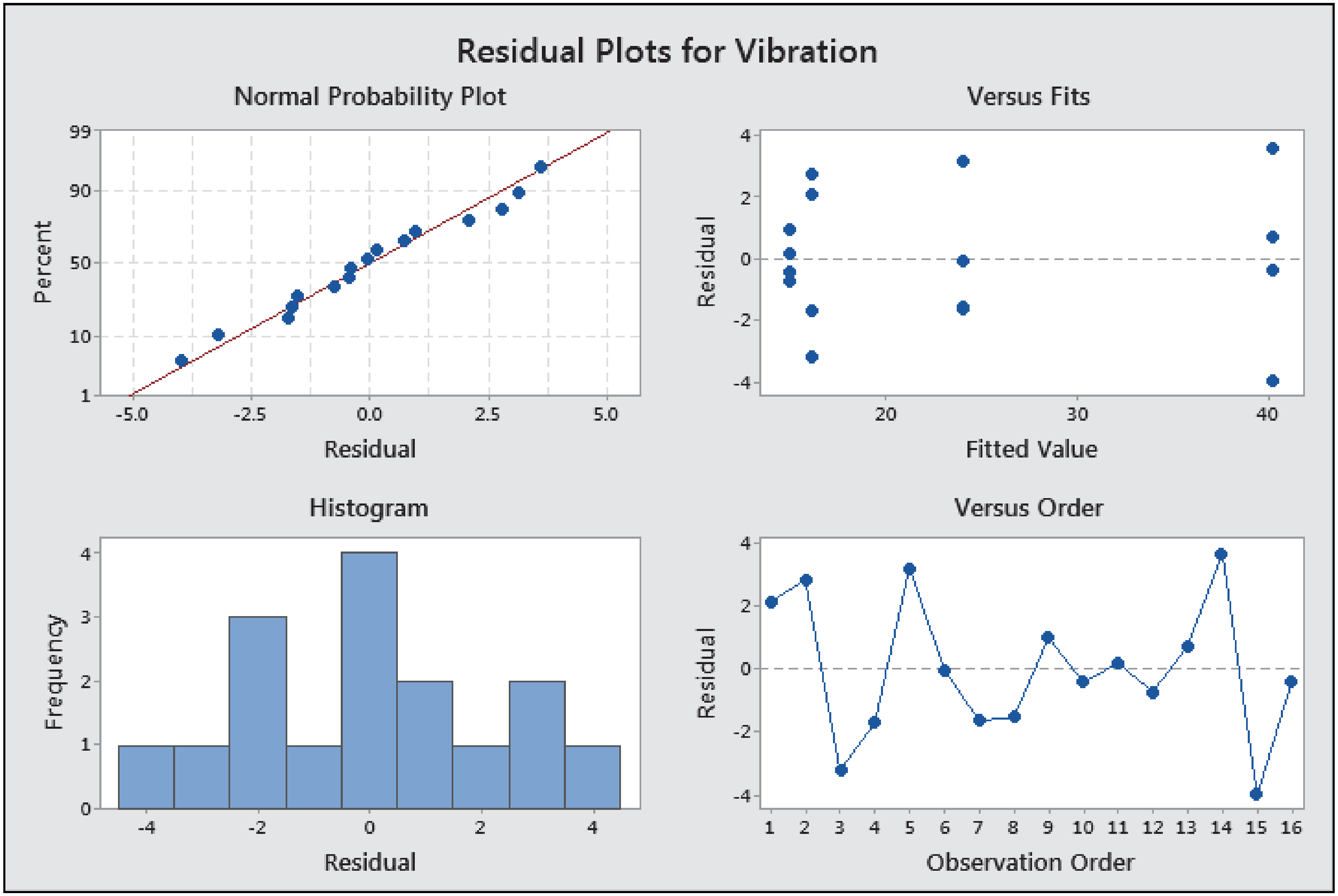
R-Code

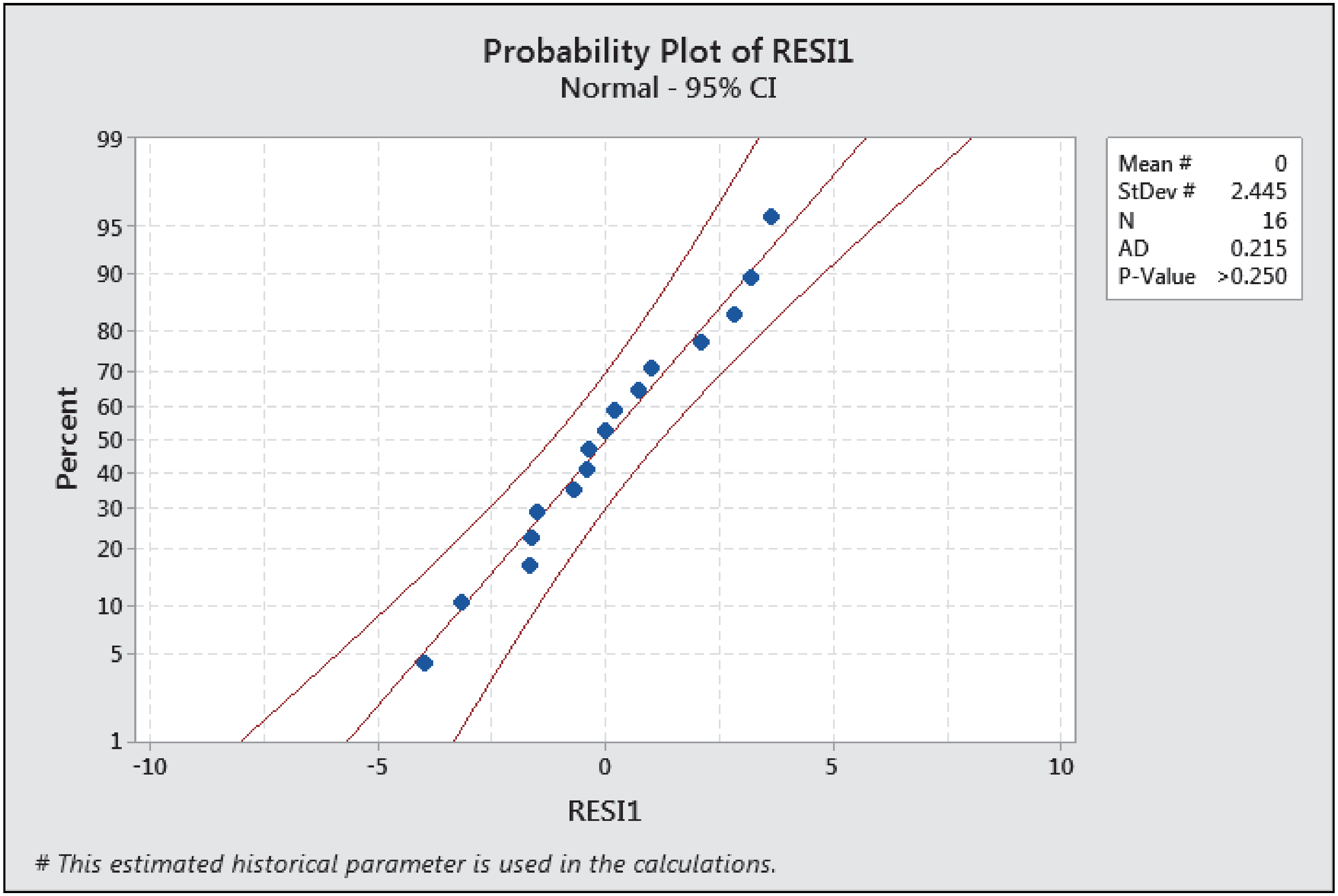
Model Summary

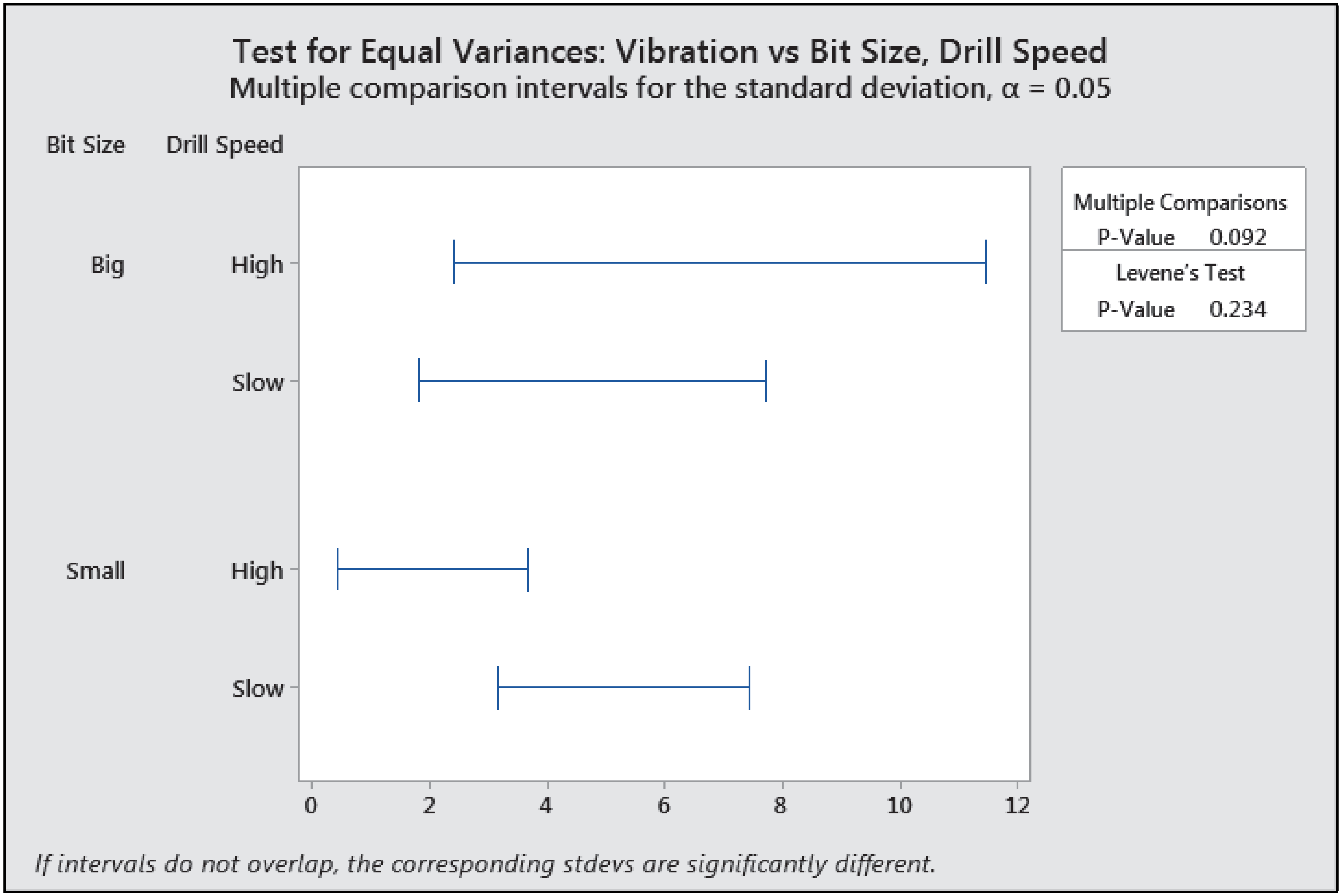
S	R-sq	R-sq(adj)
2.44476	95.81%	94.76%

```
> R_squared_adj<-(1-MS_E/MS_T)
> R_squared_adj
[1] 0.9475662
```


Interaction Plot for Vibration
Data Means
Drill Speed
High
Slow
Mean
40
35
30
25
20
15
Big
Small
Bit Size


Residual Plots for Vibration
Normal Probability Plot
Percent
99
90
50
10
1
-5.0
-2.5
0.0
2.5
5.0
Residual
Versus Fits
Residual
4
2
0
-2
-4
20
30
40
Fitted Value
Histogram
Frequency
4
3
2
1
0
-4
-2
0
2
4
Residual
Versus Order
Residual
4
2
0
-2
-4
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
Observation Order


Probability Plot of RESI1
Normal - 95% CI
Percent
99
95
90
80
70
60
50
40
30
20
10
5
1
-10
-5
0
5
10
RESI1
Mean # 0
StDev # 2.445
N 16
AD 0.215
P-Value >0.250
This estimated historical parameter is used in the calculations.


Test for Equal Variances: Vibration vs Bit Size, Drill Speed
Multiple comparison intervals for the standard deviation, α = 0.05
Bit Size
Drill Speed
Big
High
Slow
Small
High
Slow
0
2
4
6
8
10
12
Multiple Comparisons
P-Value 0.092
Levene's Test
P-Value 0.234
If intervals do not overlap, the corresponding stdevs are significantly different.

2^4 DESIGN OF EXPERIMENT EXAMPLE:

An article in *Solid State Technology ("Orthogonal Design for Process Optimization and Its Application in Plasma Etching,"* May 1987, pp. 1227-132) describes the application of factorial design in developing a nitride etch process on a single wafer plasma etcher. The process uses C_2F_6 as the reactant gas. It is possible to vary the gas flow, the power applied to the cathode, the pressure in the reactor chamber, and the spacing between the anode and the cathode (gap). Several response variables would usually be of interest in this process, but in this example we will concentrate on etch rate for silicon nitride. We will use a single replicate of a 2^4 design to investigate this process. The factor levels used in the design are shown below.

DESIGN FACTORS

	Gap	$Pressure$	$C_2F_6\ Flow$	$Power$
	A	B	C	D
$Level$	(cm)	$(mTorr)$	$(SCCM)$	(W)
$Low\,(-)$	0.80	450	125	275
$High\,(+)$	1.20	550	200	325

The Test Data and accompanying ANOVA Analysis is presented in the spreadsheet entitled "Etch_Example.xls"

Analysis in file "Etching_Experiment.R"

```
# A tibble: 16 x 3
   Scenarios  `1`   sum
   <chr>     <dbl> <dbl>
 1 (1)         550   550
 2 a           669   669
 3 b           604   604
 4 ab          650   650
 5 c           633   633
 6 ac          642   642
 7 bc          601   601
 8 abc         635   635
 9 d          1037  1037
10 ad          749   749
11 bd         1052  1052
12 abd         868   868
13 cd         1075  1075
14 acd         860   860
15 bcd        1063  1063
16 abcd        729   729
```

	A	B	C	D	AB	AC	BC	ABC	AD	BD	ABD	CD	ACD	BCD	ABCD
(1)	-1	-1	-1	-1	1	1	1	-1	1	1	-1	1	-1	-1	1
a	1	-1	-1	-1	-1	-1	1	1	-1	1	1	1	1	-1	-1
b	-1	1	-1	-1	-1	1	-1	1	1	-1	1	1	-1	1	-1
ab	1	1	-1	-1	1	-1	-1	-1	-1	-1	-1	1	1	1	1
c	-1	-1	1	-1	1	-1	-1	1	1	1	-1	-1	1	1	-1
ac	1	-1	1	-1	-1	1	-1	-1	-1	1	1	-1	-1	1	1
bc	-1	1	1	-1	-1	-1	1	-1	1	-1	1	-1	1	-1	1
abc	1	1	1	-1	1	1	1	1	-1	-1	-1	-1	-1	-1	-1
d	-1	-1	-1	1	1	1	1	-1	-1	-1	1	-1	1	1	-1
ad	1	-1	-1	1	-1	-1	1	1	1	-1	-1	-1	-1	1	1
bd	-1	1	-1	1	-1	1	-1	1	-1	1	-1	-1	1	-1	1
abd	1	1	-1	1	1	-1	-1	-1	1	1	1	-1	-1	-1	-1
cd	-1	-1	1	1	1	-1	-1	1	-1	-1	1	1	-1	-1	1
acd	1	-1	1	1	-1	1	-1	-1	1	-1	-1	1	1	-1	-1
bcd	-1	1	1	1	-1	-1	1	-1	-1	1	-1	1	-1	1	-1
abcd	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Analysis in file "Etching_Experiment.R"

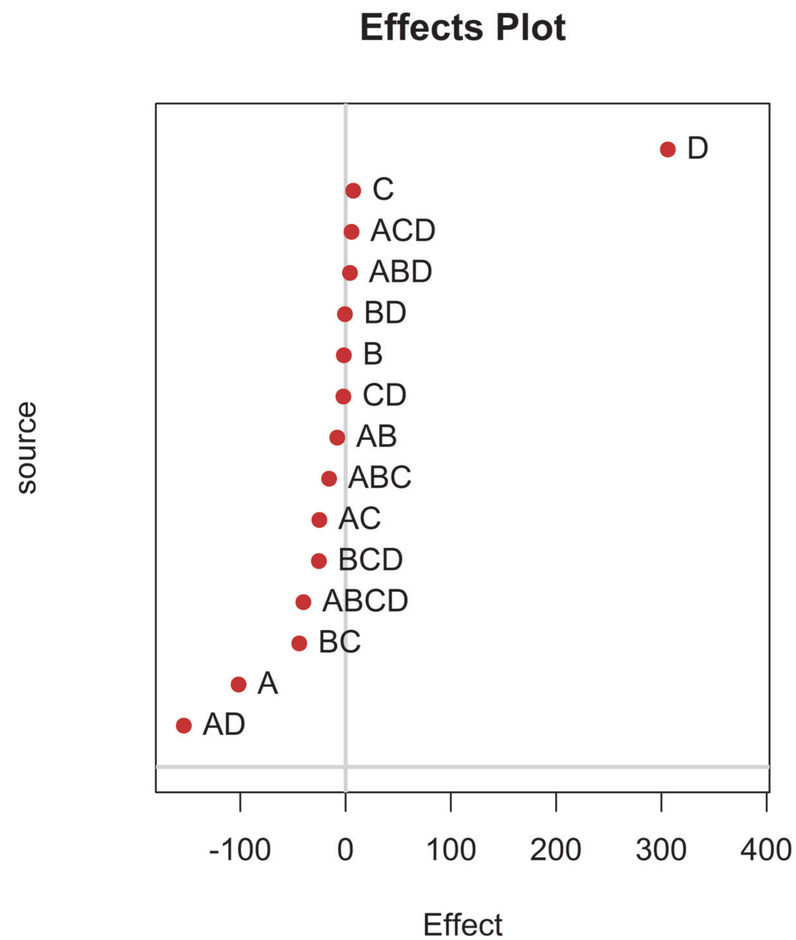
	on_off	contrasts	SS	effects	coeff
A	1	-813	41310.5625	-101.625	-50.8125
B	1	-13	10.5625	-1.625	-0.8125
C	1	59	217.5625	7.375	3.6875
D	1	2449	374850.0625	306.125	153.0625
AB	1	-63	248.0625	-7.875	-3.9375
AC	1	-199	2475.0625	-24.875	-12.4375
BC	1	-351	7700.0625	-43.875	-21.9375
ABC	1	-125	976.5625	-15.625	-7.8125
AD	1	-1229	94402.5625	-153.625	-76.8125
BD	1	-5	1.5625	-0.625	-0.3125
ABD	1	33	68.0625	4.125	2.0625
CD	1	-17	18.0625	-2.125	-1.0625
ACD	1	45	126.5625	5.625	2.8125
BCD	1	-203	2575.5625	-25.375	-12.6875
ABCD	1	-321	6440.0625	-40.125	-20.0625

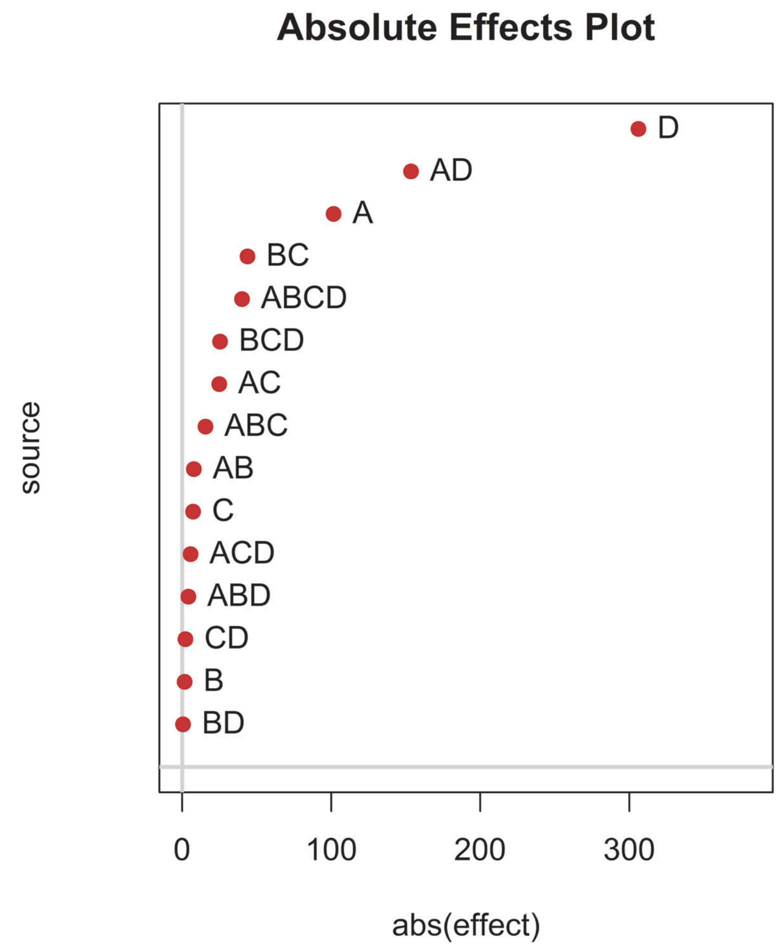
Analysis in file "Etching_Experiment.R"

```
# A tibble: 16 x 2
    pred `res 1`
   <dbl>   <dbl>
 1   550       0
 2   669       0
 3   604       0
 4   650       0
 5   633       0
 6   642       0
 7   601       0
 8   635       0
 9  1037       0
10   749       0
11  1052       0
12   868       0
13  1075       0
14   860       0
15  1063       0
16   729       0
```

SOURCE	SS	df	MS	F-value	p-value
A	41310.56	1	41310.56		
B	10.56	1	10.56		
C	217.56	1	217.56		
D	374850.06	1	374850.06		
AB	248.06	1	248.06		
AC	2475.06	1	2475.06		
BC	7700.06	1	7700.06		
ABC	976.56	1	976.56		
AD	94402.56	1	94402.56		
BD	1.56	1	1.56		
ABD	68.06	1	68.06		
CD	18.06	1	18.06		
ACD	126.56	1	126.56		
BCD	2575.56	1	2575.56		
ABCD	6440.06	1	6440.06		
Total	531420.94	15			

Analysis in file "Etching_Experiment.R"



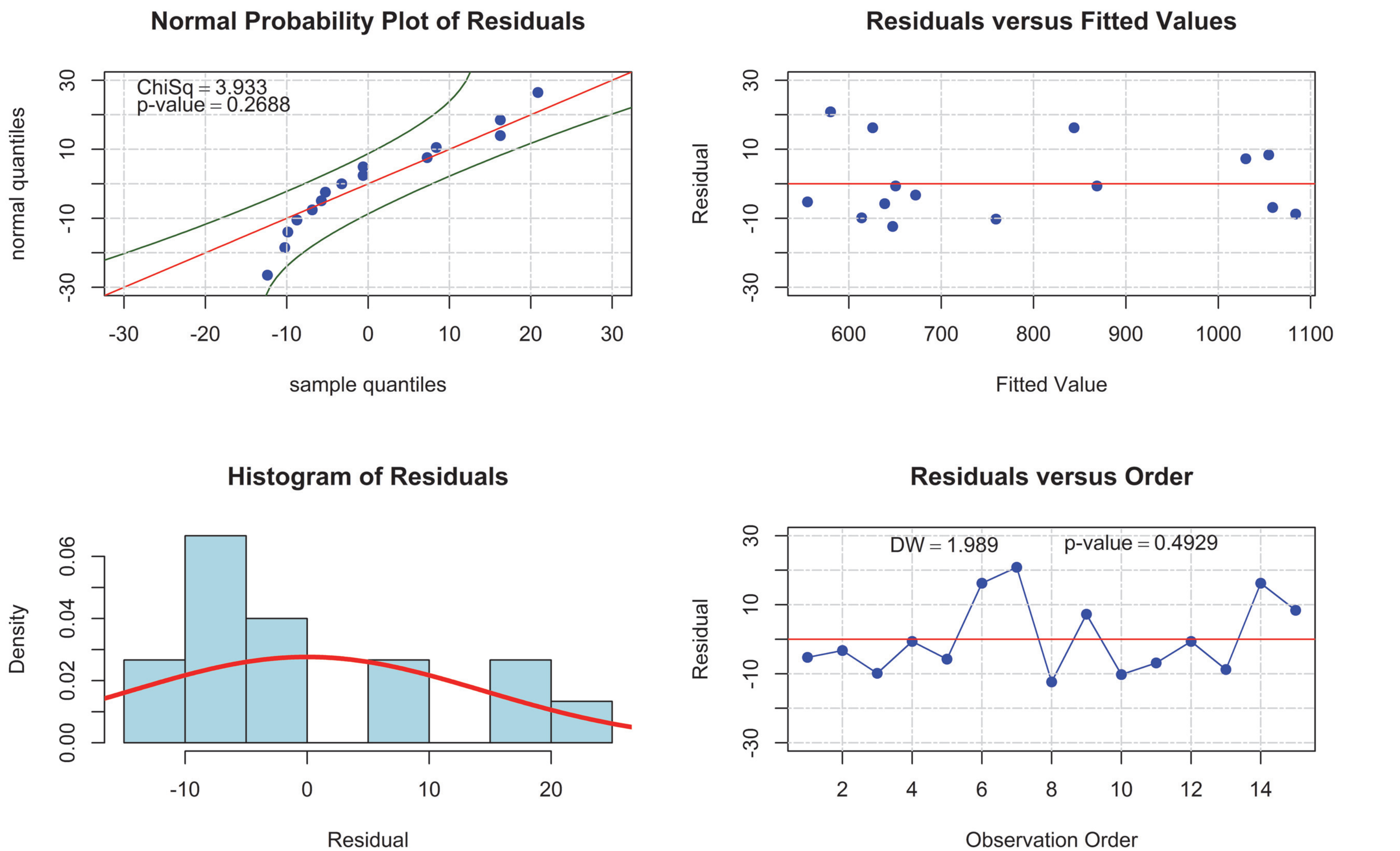


Turning off **8 Effects**: BD, B, CD, ABD, ACD,C, AB, ABC

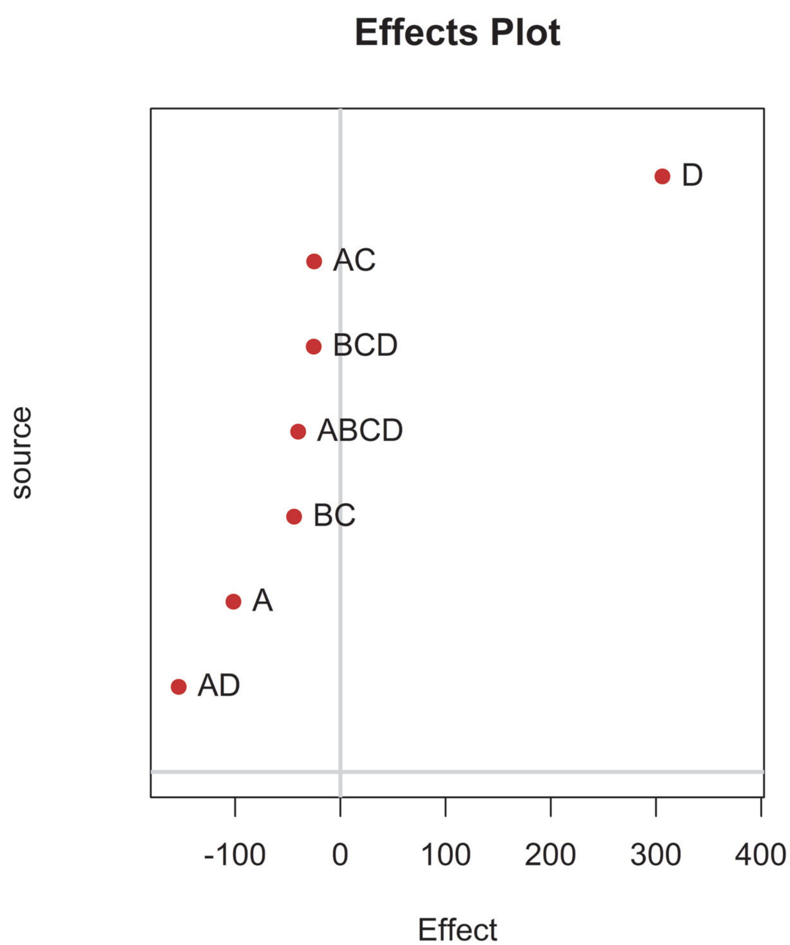
```
# A tibble: 16 x 2
    pred `res 1`
   <dbl>   <dbl>
 1  717. -167.
 2  419.  250.
 3 1158. -554.
 4  640.    9.62
 5  846. -213.
 6  548.   94.1
 7 1287. -686.
 8  769. -134.
 9  538.  499.
10  787.  -38.4
11  894.  158.
12 1155. -287.
13  287.  788.
14  842.   18.4
15  889.  174.
16  642.   87.1
```

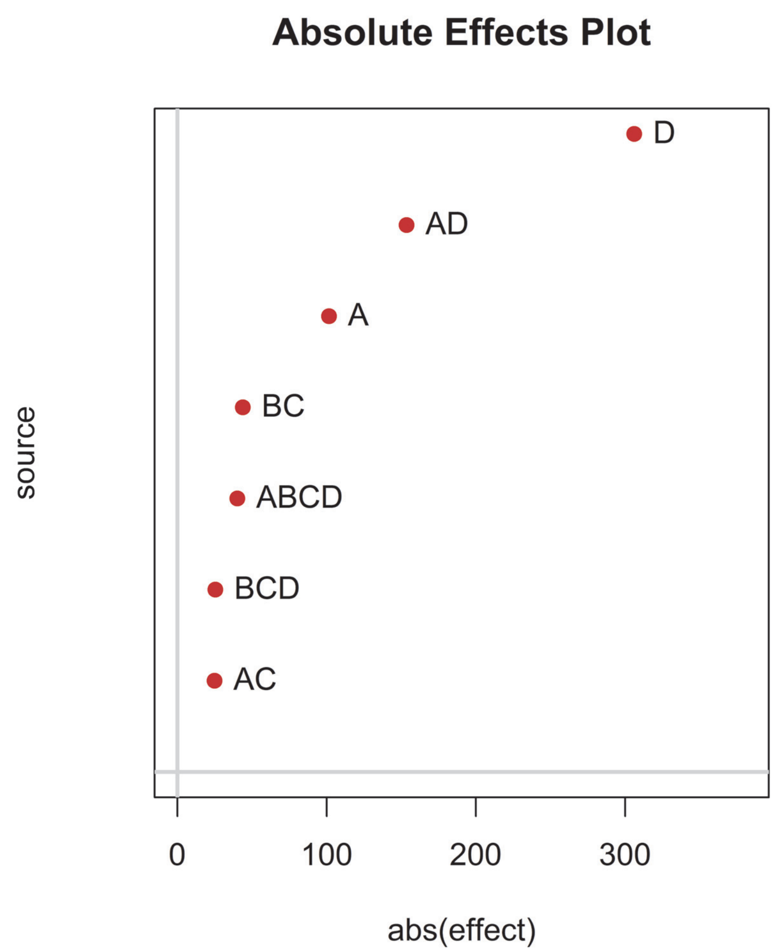
SOURCE	SS	df	MS	F-value	p-value
A	41310.56	1	41310.56	198.25	0.00 %
D	374850.06	1	374850.06	1798.92	0.00 %
AC	2475.06	1	2475.06	11.88	0.87 %
BC	7700.06	1	7700.06	36.95	0.03 %
AD	94402.56	1	94402.56	453.04	0.00 %
BCD	2575.56	1	2575.56	12.36	0.79 %
ABCD	6440.06	1	6440.06	30.91	0.05 %
Error	1667.00	8	208.38		
Total	531420.94	15			

2^K FACTORIAL ANOVA — Etching Model Diagnostics



Analysis in file "Etching_Experiment.R"

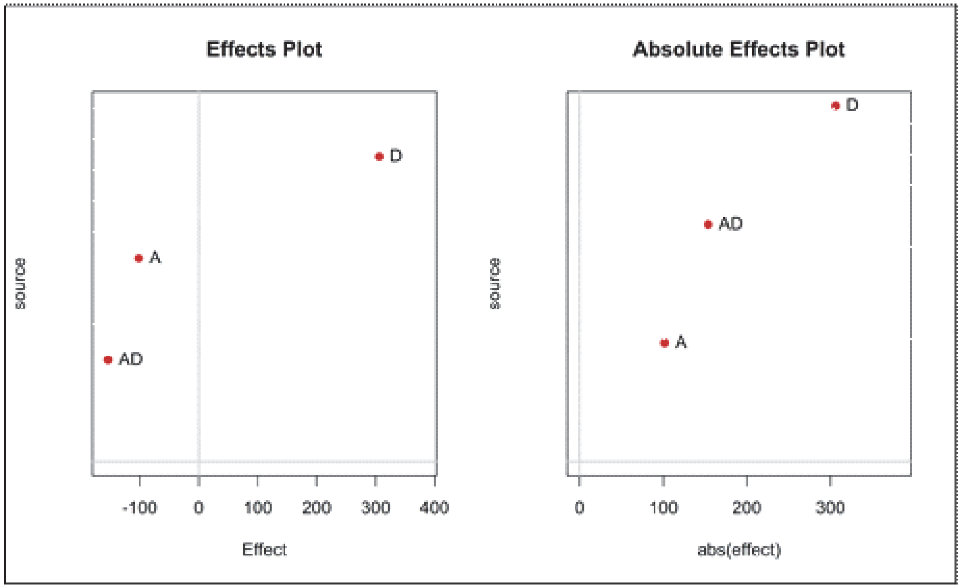


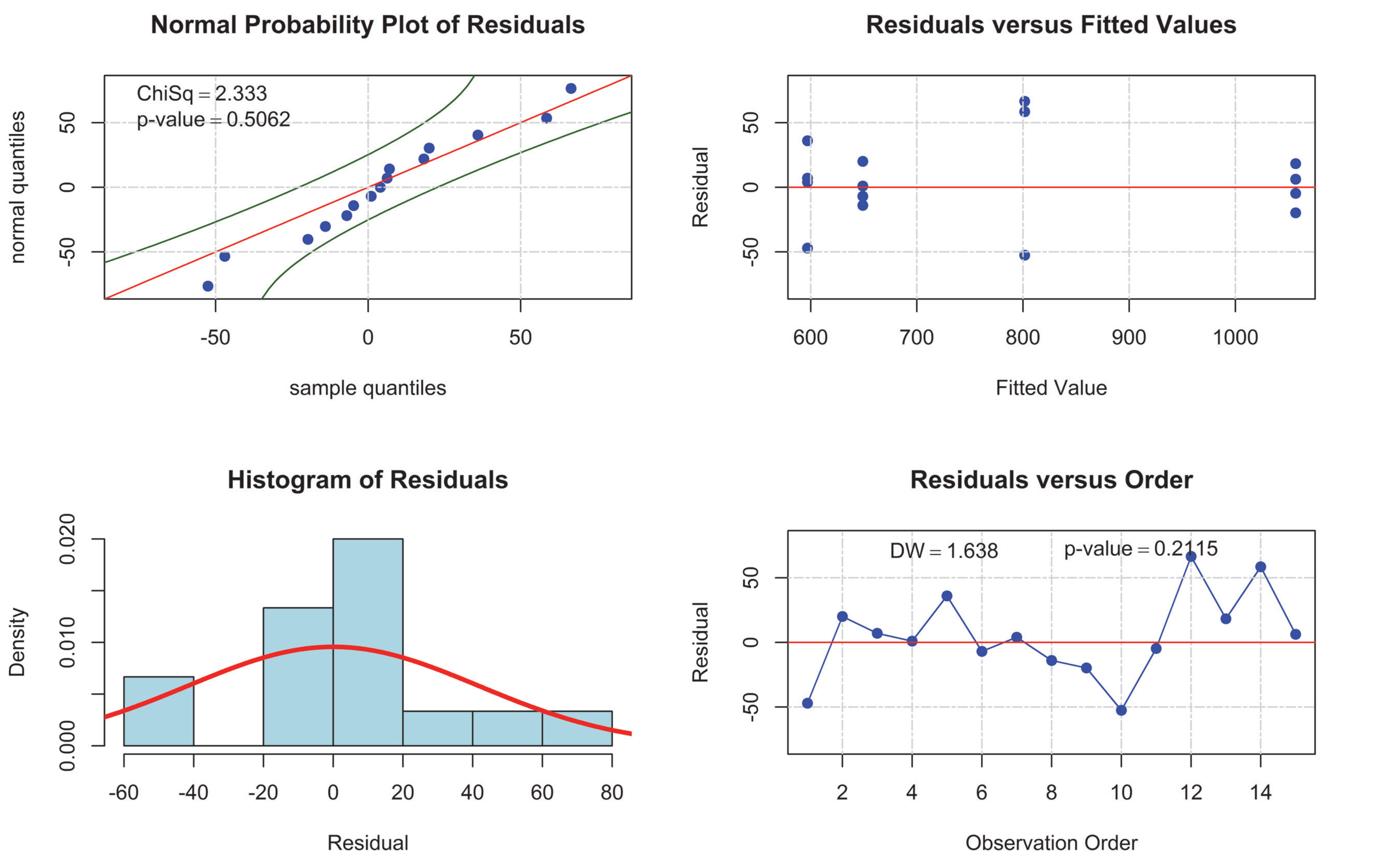


Turning off **4 Additional Effects**: AC, BCD, ABCD, BC

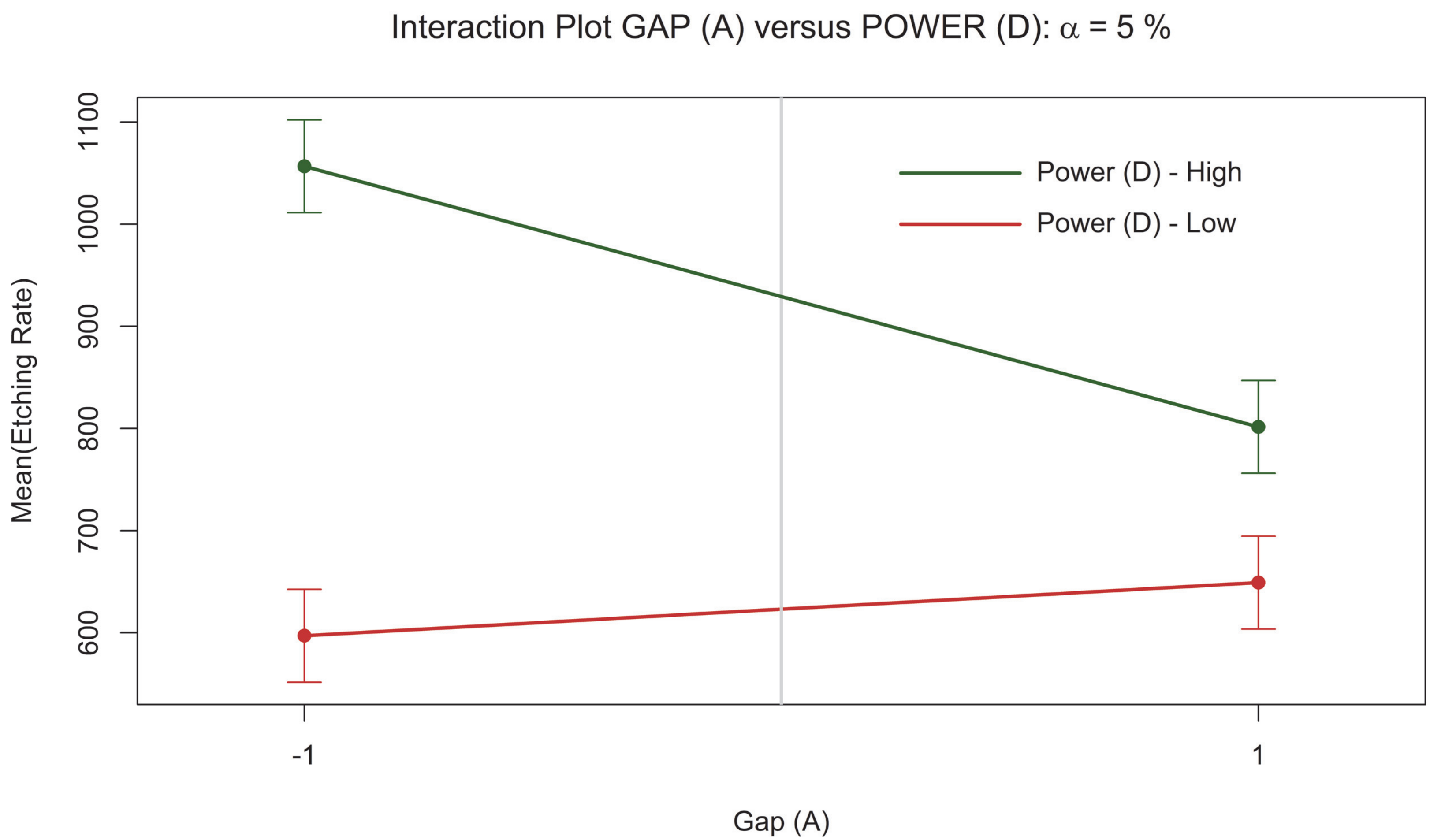
```
# A tibble: 16 x 2
    pred `res 1`
   <dbl>   <dbl>
 1   597  -47
 2   649   20
 3   597    7
 4   649    1
 5   597   36
 6   649   -7
 7   597    4
 8   649  -14
 9 1057.  -19.8
10  802.  -52.5
11 1057.   -4.75
12  802.   66.5
13 1057.   18.2
14  802.   58.5
15 1057.    6.25
16  802.  -72.5
```

SOURCE	SS	df	MS	F-value	p-value
A	41310.56	1	41310.56	23.77	0.04 %
D	374850.06	1	374850.06	215.66	0.00 %
AD	94402.56	1	94402.56	54.31	0.00 %
Error	20857.75	12	1738.15		
Total	531420.94	15			





Normal Probability Plot of Residuals
ChiSq = 2.333
p-value = 0.5062
normal quantiles
sample quantiles
-50
0
50
Residuals versus Fitted Values
Residual
Fitted Value
600
700
800
900
1000
Histogram of Residuals
Density
Residual
0.000
0.010
0.020
-60
-40
-20
0
20
40
60
80
Residuals versus Order
DW = 1.638
p-value = 0.2115
Residual
Observation Order
2
4
6
8
10
12
14



Conclusion: Choose a **LOW GAP (A)** and **HIGH POWER (D)**, **confidence intervals do not overlap** (and one observes the interaction)!